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Journal of Computational and Applied Mathematics 106 (1999) 369–393

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.nl/locate/cam

Differential operators having symmetric orthogonal polynomials as eigenfunctions

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Received 25 May 1998; received in revised form 27 February 1999

Abstract

Let the polynomials $\{P_n(x)\}_{n=0}^{\infty}$, orthogonal with respect to a symmetric positive definite moment functional σ, be eigenfunctions of a linear differential operator $\boldsymbol{L}$. We consider the orthogonal polynomials $\{P_n^{\mu}(x)\}_{n=0}^{\infty}$ and $\{P_n^{\mu,\nu}(x)\}_{n=0}^{\infty}$, which are obtained by adding one resp. two symmetric (Sobolev type) terms to σ. In all the cases we derive a representation for the polynomials and show that they are eigenfunctions of one or more linear differential operators (mostly of infinite order) of the form $\boldsymbol{L}+\mu\boldsymbol{A}$ resp. $\boldsymbol{L}+\mu\boldsymbol{A}+\nu\boldsymbol{B}+\mu\nu\boldsymbol{C}$. Further it is investigated to what extend the eigenvalues can be chosen arbitrarily and finally expressions are given for the other eigenvalues. © 1999 Elsevier Science B.V. All rights reserved.

MSC: 34A35; 33C45

Keywords: Differential operator; Orthogonal polynomial

1. Introduction

Let $\mathfrak{P}$ be the space of all real polynomials and let $\{P_n(x)\}_{n=0}^{\infty}$ be a system of polynomials with $\deg[P_n(x)]=n$ for each n, which are eigenfunctions of a linear differential operator $\boldsymbol{L}=\boldsymbol{L}(x)$ mapping $\mathfrak{P}$ into $\mathfrak{P}$ with eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$, which are not all zero. Hence

$$\boldsymbol{L}P_n(x)=\lambda_n P_n(x) \quad \text{for all } n\in\{0,1,2,\ldots\}.$$

In [1] a system of polynomials $\{P_n^{\mu}(x)\}_{n=0}^{\infty}=\{P_n(x)+\mu Q_n(x)\}_{n=0}^{\infty}$ was considered, where μ is a real parameter and $\{Q_n(x)\}_{n=0}^{\infty}$ denotes a system of polynomials with deg $[Q_n(x)]\leqslant n$. The object was to find linear differential operators $\boldsymbol{A}$ mapping $\mathfrak{P}$ into $\mathfrak{P}$ and numbers $\{\alpha_n\}_{n=0}^{\infty}$ such that the polynomials $\{P_n^{\mu}(x)\}_{n=0}^{\infty}$ (linear perturbations of $\{P_n(x)\}_{n=0}^{\infty}$) are eigenfunctions of the linear operator

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PII: S0377-0427(99)00094-1

$\boldsymbol{L}+\mu\boldsymbol{A}$ (a linear perturbation of $\boldsymbol{L}$) with eigenvalues $\{\lambda_n+\mu\alpha_n\}_{n=0}^{\infty}$ (linear perturbations of $\{\lambda_n\}_{n=0}^{\infty}$), i.e.

$$[(\boldsymbol{L}-\lambda_n\boldsymbol{I})+\mu(\boldsymbol{A}-\alpha_n\boldsymbol{I})]P_n^{\mu}(x)=0, \quad n=0,1,2,\ldots. \tag{1}$$

Here $\boldsymbol{I}$ denotes the identity operator. Necessary and sufficient conditions for the polynomials $\{Q_n(x)\}_{n=0}^{\infty}$ are derived such that operators $\boldsymbol{A}$ mapping $\mathfrak{P}$ into $\mathfrak{P}$ and numbers $\{\alpha_n\}_{n=0}^{\infty}$ exist. The main applications in [1] (see also [2]) were found in the polynomials, which are orthogonal with respect to an inner product, originating from an inner product for the classical orthogonal polynomials, by adding a discrete term, possibly of Sobolev type. Three different kinds of discrete terms added to the inner product were studied, the first of them concerning the general case of classical orthogonal polynomials and the other two concerning classical orthogonal polynomials with respect to a symmetric inner product (Gegenbauer and Hermite polynomials). In this paper are considered three cases of adding two different symmetric terms to a symmetric inner product. In a way similar to [3] representations of the polynomials are found and moreover in each case it is shown that the polynomials are eigenfunctions of linear differential operators of the form $\boldsymbol{L}+\mu\boldsymbol{A}+\nu\boldsymbol{B}+\mu\nu\boldsymbol{C}$, it is investigated to what extend the eigenvalues can be chosen arbitrarily and finally expressions are given for the other eigenvalues. Since by the results of this paper the existence of such a linear differential operator (mostly of infinite order) is guaranteed, it is a challenge to actually compute the eigenvalues and the coefficients of the differential operator, certainly if this operator turns out to be of finite order in some cases. Examples will be given in the last section.

2. Symmetric and special perturbations

2.1. Orthogonal polynomials and differential operator

Let σ be a symmetric positive-definite moment functional (see [7], Definition I.4.1) and let $\{P_n(x)\}_{n=0}^{\infty}$ with deg $[P_n(x)]=n$ be the polynomials orthogonal with respect to σ. Then for all x and all $n\in\{0,1,2,\ldots\}$ we have $P_n(-x)=(-1)^nP_n(x)$. Further let $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of real numbers with $\lambda_0=0$ and $\{\lambda_n\}_{n=1}^{\infty}$ not all equal to zero such that $\{P_n(x)\}_{n=0}^{\infty}$ is a polynomial set of solutions of

$$\boldsymbol{L}y(x)\equiv\sum_{i=1}^{\infty}e_i(x)y^{(i)}(x)=\lambda_ny(x). \tag{2}$$

Here $\{e_i(x)\}_{i=1}^{\infty}$ is a sequence of polynomials with $\deg[e_i(x)]\leqslant i$ for all $i\in\mathbb{N}\setminus\{0\}$, called the coefficients of $\boldsymbol{L}$.

2.2. Symmetric perturbation

Let ϕ be the symmetric bilinear form (of Sobolev type if $l>0$) defined by

$$\phi(p,q)=\langle\sigma,pq\rangle+\mu[p^{(l)}(c)q^{(l)}(c)+p^{(l)}(-c)q^{(l)}(-c)],$$

where $p,q\in\mathfrak{P}$, $\mu>0$, $c>0$ and $l\in\{0,1,2,\ldots\}$. The polynomials $\{P_n^{\mu}(x)\}_{n=0}^{\infty}$, orthogonal with respect to ϕ, can be written as

$$P_n^{\mu}(x)=P_n(x)+\mu Q_n(x) \tag{3}$$

with

$$Q_n(x) = \sum_{k=0}^{n} q_{nk} P_k(x). \tag{4}$$

By using the notation

$$K_n^{(r,s)}(x,y) = \sum_{k=0}^{n} \frac{P_k^{(r)}(x) P_k^{(s)}(y)}{\langle \sigma, P_k^2 \rangle}, \quad n,r,s \in \{0,1,2,\ldots\},$$

we have (see [2])

$$P_n^\mu(x) = [1 + \mu\{K_{n-1}^{(l,l)}(c,c) + (-1)^{n+l} K_{n-1}^{(l,l)}(c,-c)\}] P_n(x)$$
$$-\mu P_n^{(l)}(c)\{K_{n-1}^{(0,l)}(x,c) + (-1)^{n+l} K_{n-1}^{(0,l)}(x,-c)\}$$

or

$$P_n^\mu(x) = P_n(x) + \mu \begin{vmatrix} P_n(x) & K_{n-1}^{(0,l)}(x,c) + (-1)^{n+l} K_{n-1}^{(0,l)}(x,-c) \\ P_n^{(l)}(c) & K_{n-1}^{(l,l)}(c,c) + (-1)^{n+l} K_{n-1}^{(l,l)}(c,-c) \end{vmatrix}. \tag{5}$$

Hence

$$q_{nn} = K_{n-1}^{(l,l)}(c,c) + (-1)^{n+l} K_{n-1}^{(l,l)}(c,-c). \tag{6}$$

Since for $k < n$

$$q_{nk} = \frac{-P_n^{(l)}(c) P_k^{(l)}(c)[1 + (-1)^{n+k}]}{\langle \sigma, P_k^2 \rangle}, \tag{7}$$

it is obvious that $q_{nk} = 0$ if $n + k$ is odd and hence for all x and all $n \in \mathbb{N}$ we have $P_n^\mu(-x) = (-1)^n P_n^\mu(x)$. We may conclude that in the case that $P_n^{(l)}(c) \neq 0$ for all $n \in \{l, l+1, l+2, \ldots\}$ the set of symmetric polynomials $\{P_n^\mu(x)\}_{n=0}^\infty$ given by (5) is a symmetric linear perturbation of $\{P_n(x)\}_{n=0}^\infty$ of the class l (see [2]). We will call $\{P_n^\mu(x)\}_{n=0}^\infty$ *the symmetric* (l,c,μ) *perturbation of* $\{P_n(x)\}_{n=0}^\infty$ and we know that there exist operators A of the form

$$Ay(x) \equiv \sum_{i=1}^{\infty} a_i(x) y^{(i)}(x), \tag{8}$$

where $\{a_i(x)\}_{i=1}^\infty$ is a sequence of polynomials with $\deg[a_i(x)] \leqslant i$ for all $i = 1,2,3,\ldots$, and sequences of real numbers $\{\alpha_n\}_{n=0}^\infty$ with $\alpha_0 = 0$ such that

$$(\boldsymbol{L} + \mu A) P_n^\mu(x) = (\lambda_n + \mu\alpha_n) P_n^\mu(x), \quad n = 0,1,2,\ldots\ . \tag{9}$$

The numbers $\{\alpha_n\}_{n=1}^{l+1}$ can be chosen arbitrarily and the values of $\{\alpha_n\}_{n=l+2}^\infty$ and the operator A are uniquely determined when these arbitrary numbers have been chosen. We have for $t \in \{1,2,3,\ldots\}$

$$\alpha_{l+2t} = \alpha_l + \sum_{j=1}^{t} (\lambda_{l+2j} - \lambda_{l+2j-2}) q_{l+2j\ l+2j}, \tag{10}$$

$$\alpha_{l+2t+1} = \alpha_{l+1} + \sum_{j=1}^{t} (\lambda_{l+2j+1} - \lambda_{l+2j-1}) q_{l+2j+1\ l+2j+1}. \tag{11}$$

2.3. Special perturbation

Let ϕ_0 be the symmetric bilinear form (of Sobolev type if $l > 0$) defined by

$$\phi_0(p,q) = \langle\sigma, pq\rangle + \mu p^{(l)}(0)\, q^{(l)}(0),$$

where $p, q \in \mathfrak{P}$, $\mu > 0$ and $l \in \{0,1,2,\ldots\}$. The polynomials $\{P_n^{\mu}(x)\}_{n=0}^{\infty}$, orthogonal with respect to ϕ_0, can be written as (3) with (4). We have (see for instance [1])

$$P_n^{\mu}(x) = [1 + \mu K_{n-1}^{(l,l)}(0,0)]P_n(x) - \mu P_n^{(l)}(0)K_{n-1}^{(0,l)}(x,0). \tag{12}$$

Hence

$$q_{nn} = K_{n-1}^{(l,l)}(0,0). \tag{13}$$

Since for $k < n$

$$q_{nk} = \frac{-P_n^{(l)}(0)P_k^{(l)}(0)}{\langle\sigma, P_k^2\rangle}, \tag{14}$$

it is obvious that $q_{nk} = 0$ if $l + k$ or $l + n$ is odd and hence for all x and all $n \in \mathbb{N}$ we have $P_n^{\mu}(-x) = (-1)^n P_n^{\mu}(x)$. We may conclude that in the case that $P_n^{(l)}(0) \neq 0$ for all $n \geqslant l$ with $n - l$ even the set of symmetric polynomials $\{P_n^{\mu}(x)\}_{n=0}^{\infty}$ given by (12) is a special linear perturbation of $\{P_n(x)\}_{n=0}^{\infty}$ of the class l (see [1]). We will call $\{P_n^{\mu}(x)\}_{n=0}^{\infty}$ *the special* (l,μ) *perturbation of* $\{P_n(x)\}_{n=0}^{\infty}$ and we know that there exist operators A of the form (8) and sequences of real numbers $\{\alpha_n\}_{n=0}^{\infty}$ with $\alpha_0=0$ such that (9) is valid. The numbers $\{\alpha_{l+2t-1}\}_{t=1}^{\infty}$ and, if $l > 0$, the numbers $\{\alpha_n\}_{n=1}^{l}$ can be chosen arbitrarily. The values of $\{\alpha_{l+2t}\}_{t=1}^{\infty}$ and the operator A are uniquely determined, when these arbitrary numbers have been chosen. We have for $t \in \{1,2,3,\ldots\}$

$$\alpha_{l+2t} = \alpha_l + \sum_{j=1}^{t} (\lambda_{l+2j} - \lambda_{l+2j-2})\, q_{l+2j\ l+2j}. \tag{15}$$

3. Two symmetric perturbations

3.1. Representation of the polynomials

In this section we consider the polynomials $\{P_n^{\mu,\nu}(x)\}_{n=0}^{\infty}$, orthogonal with respect to the symmetric bilinear form (of Sobolev type) defined by

$$\begin{aligned}\psi_1(p,q) = {} & \langle\sigma, pq\rangle + \mu[p^{(l_1)}(c_1)q^{(l_1)}(c_1) + p^{(l_1)}(-c_1)q^{(l_1)}(-c_1)] \\ & + \nu[p^{(l_2)}(c_2)q^{(l_2)}(c_2) + p^{(l_2)}(-c_2)q^{(l_2)}(-c_2)],\end{aligned} \tag{16}$$

where $p, q \in \mathfrak{P}$, $\mu > 0$, $\nu > 0$, $c_1 > 0$ and $c_2 > 0$, $l_1, l_2 \in \{0,1,2,\ldots\}$ and $(c_1, l_1) \neq (c_2, l_2)$. Without loss of the generality we may assume

$$l_1 \leqslant l_2.$$

We construct these polynomials by first taking the symmetric (l_1, c_1, μ) perturbation of $\{P_n(x)\}_{n=0}^{\infty}$, thus obtaining the polynomials $\{P_n^{\mu}(x)\}_{n=0}^{\infty}$ orthogonal with respect to

$$\phi_1(p,q) = \langle\sigma, pq\rangle + \mu[p^{(l_1)}(c_1)q^{(l_1)}(c_1) + p^{(l_1)}(-c_1)q^{(l_1)}(-c_1)].$$

Afterwards the symmetric (l_2, c_2, ν) perturbation of $\{P_n^{\mu}(x)\}_{n=0}^{\infty}$ is taken. By Section 2.2 we can write

$$P_n^{\mu}(x) = [1 + \mu\{K_{n-1}^{(l_1,l_1)}(c_1,c_1) + (-1)^{n+l_1}K_{n-1}^{(l_1,l_1)}(c_1,-c_1)\}]P_n(x)$$
$$-\mu P_n^{(l_1)}(c_1)\{K_{n-1}^{(0,l_1)}(x,c_1) + (-1)^{n+l_1}K_{n-1}^{(0,l_1)}(x,-c_1)\}. \tag{17}$$

In a way similar to [8], Proposition 3.2, in the next proposition a formula is derived for the kernel

$$G_n(x,y;\mu) = \sum_{i=0}^{n} \frac{P_i^{\mu}(x)P_i^{\mu}(y)}{\phi_1(P_i^{\mu},P_i^{\mu})}. \tag{18}$$

Proposition 1.

$$G_n(x,y;\mu) + (-1)^m G_n(x,-y;\mu)$$
$$= K_n(x,y) + (-1)^m K_n(x,-y)$$
$$-\mu\frac{\{K_n^{(0,l_1)}(x,c_1) + (-1)^{m+l_1}K_n^{(0,l_1)}(x,-c_1)\}\{K_n^{(l_1,0)}(c_1,y) + (-1)^m K_n^{(l_1,0)}(c_1,-y)\}}{1 + \mu\{K_n^{(l_1,l_1)}(c_1,c_1) + (-1)^{m+l_1}K_n^{(l_1,l_1)}(c_1,-c_1)\}},$$
$$(n, m \in \{0,1,2,\ldots\}). \tag{19}$$

Proof. It is well known that the kernel $G_n(x,y;\mu)$ has the reproducing property

$$\phi_1(G_n(x,y;\mu), q(x)) = q(y)$$

for any polynomial $q(x)$ of degree $\leqslant n$. If we put

$$G_n(x,y;\mu) = \sum_{i=0}^{n} C_{n,i}(y;\mu)P_i(x)$$

then

$$\phi_1(G_n(x,y;\mu), P_k(x)) = P_k(y) = \left\langle \sigma, \sum_{i=0}^{n} C_{n,i}(y;\mu)P_i(x)P_k(x) \right\rangle$$
$$+\mu[G_n^{(l_1,0)}(c_1,y;\mu)P_k^{(l_1)}(c_1) + G_n^{(l_1,0)}(-c_1,y;\mu)P_k^{(l_1)}(-c_1)].$$

It follows that

$$C_{n,k}(y;\mu) = \frac{P_k(y)}{\langle\sigma, P_k^2\rangle} - \mu\frac{[G_n^{(l_1,0)}(c_1,y;\mu)P_k^{(l_1)}(c_1) + G_n^{(l_1,0)}(-c_1,y;\mu)P_k^{(l_1)}(-c_1)]}{\langle\sigma, P_k^2\rangle}$$

which leads to

$$G_n(x,y;\mu) = K_n(x,y) - \mu[G_n^{(l_1,0)}(c_1,y;\mu)K_n^{(0,l_1)}(x,c_1) + G_n^{(l_1,0)}(-c_1,y;\mu)K_n^{(0,l_1)}(x,-c_1)]. \tag{20}$$

Then

$$G_n(x,y;\mu) + (-1)^m G_n(x,-y;\mu)$$
$$= K_n(x,y) + (-1)^m K_n(x,-y)$$
$$-\mu K_n^{(0,l_1)}(x,c_1)\{G_n^{(l_1,0)}(c_1,y;\mu) + (-1)^m G_n^{(l_1,0)}(c_1,-y;\mu)\}$$
$$-\mu K_n^{(0,l_1)}(x,-c_1)\{G_n^{(l_1,0)}(-c_1,y;\mu) + (-1)^m G_n^{(l_1,0)}(-c_1,-y;\mu)\}.$$

If we notice that $G_n^{(l_1,0)}(-c_1,-y;\mu)=(-1)^{l_1}G_n^{(l_1,0)}(c_1,y;\mu)$ and hence that $G_n^{(l_1,0)}(-c_1,y;\mu)=(-1)^{l_1}G_n^{(l_1,0)}(c_1,-y;\mu)$, we find

$$\begin{aligned}
&G_n(x,y;\mu)+(-1)^m G_n(x,-y;\mu)\\
&\quad=K_n(x,y)+(-1)^m K_n(x,-y)\\
&\qquad-\mu\{G_n^{(l_1,0)}(c_1,y;\mu)+(-1)^m G_n^{(l_1,0)}(c_1,-y;\mu)\}\\
&\qquad\times\{K_n^{(0,l_1)}(x,c_1)+(-1)^{m+l_1}K_n^{(0,l_1)}(x,-c_1)\}.
\end{aligned}\tag{21}$$

When we differentiate l_1 times with respect to x and then substitute $x=c_1$ we get

$$\begin{aligned}
&G_n^{(l_1,0)}(c_1,y;\mu)+(-1)^m G_n^{(l_1,0)}(c_1,-y;\mu)\\
&\quad=K_n^{(l_1,0)}(c_1,y)+(-1)^m K_n^{(l_1,0)}(c_1,-y)\\
&\qquad-\mu\{G_n^{(l_1,0)}(c_1,y;\mu)+(-1)^m G_n^{(l_1,0)}(c_1,-y;\mu)\}\\
&\qquad\times\{K_n^{(l_1,l_1)}(c_1,c_1)+(-1)^{m+l_1}K_n^{(l_1,l_1)}(c_1,-c_1)\}.
\end{aligned}$$

Hence

$$G_n^{(l_1,0)}(c_1,y;\mu)+(-1)^m G_n^{(l_1,0)}(c_1,-y;\mu)=\frac{K_n^{(l_1,0)}(c_1,y)+(-1)^m K_n^{(l_1,0)}(c_1,-y)}{1+\mu\{K_n^{(l_1,l_1)}(c_1,c_1)+(-1)^{m+l_1}K_n^{(l_1,l_1)}(c_1,-c_1)\}}$$

and from (21) we obtain (19). □

Now we take the symmetric (l_2,c_2,ν) perturbation of $\{P_n^{\mu}(x)\}_{n=0}^{\infty}$. We obtain

$$\begin{aligned}
P_n^{\mu,\nu}(x)&=[1+\nu\{G_{n-1}^{(l_2,l_2)}(c_2,c_2;\mu)+(-1)^{n+l_2}G_{n-1}^{(l_2,l_2)}(c_2,-c_2;\mu)\}]P_n^{\mu}(x)\\
&\quad-\nu P_n^{\mu(l_2)}(c_2)\{G_{n-1}^{(0,l_2)}(x,c_2;\mu)+(-1)^{n+l_2}G_{n-1}^{(0,l_2)}(x,-c_2;\mu)\}\\
&=\Bigg[1+\nu\{K_{n-1}^{(l_2,l_2)}(c_2,c_2)+(-1)^{n+l_2}K_{n-1}^{(l_2,l_2)}(c_2,-c_2)\}\\
&\quad-\mu\nu\frac{\{K_{n-1}^{(l_2,l_1)}(c_2,c_1)+(-1)^{n+l_1}K_{n-1}^{(l_2,l_1)}(c_2,-c_1)\}\{K_{n-1}^{(l_1,l_2)}(c_1,c_2)+(-1)^{n+l_2}K_{n-1}^{(l_1,l_2)}(c_1,-c_2)\}}{1+\mu\{K_{n-1}^{(l_1,l_1)}(c_1,c_1)+(-1)^{n+l_1}K_{n-1}^{(l_1,l_1)}(c_1,-c_1)\}}\Bigg]\\
&\quad\times[(1+\mu\{K_{n-1}^{(l_1,l_1)}(c_1,c_1)+(-1)^{n+l_1}K_{n-1}^{(l_1,l_1)}(c_1,-c_1)\})P_n(x)\\
&\quad-\mu P_n^{(l_1)}(c_1)\{K_{n-1}^{(0,l_1)}(x,c_1)+(-1)^{n+l_1}K_{n-1}^{(0,l_1)}(x,-c_1)\}]\\
&\quad-\nu[(1+\mu\{K_{n-1}^{(l_1,l_1)}(c_1,c_1)+(-1)^{n+l_1}K_{n-1}^{(l_1,l_1)}(c_1,-c_1)\})P_n^{(l_2)}(c_2)\\
&\quad-\mu P_n^{(l_1)}(c_1)\{K_{n-1}^{(l_2,l_1)}(c_2,c_1)+(-1)^{n+l_1}K_{n-1}^{(l_2,l_1)}(c_2,-c_1)\}]\\
&\quad\times\Bigg[\{K_{n-1}^{(0,l_2)}(x,c_2)+(-1)^{n+l_2}K_{n-1}^{(0,l_2)}(x,-c_2)\}\\
&\quad-\mu\frac{\{K_{n-1}^{(0,l_1)}(x,c_1)+(-1)^{n+l_1}K_{n-1}^{(0,l_1)}(x,-c_1)\}\{K_{n-1}^{(l_1,l_2)}(c_1,c_2)+(-1)^{n+l_2}K_{n-1}^{(l_1,l_2)}(c_1,-c_2)\}}{1+\mu\{K_{n-1}^{(l_1,l_1)}(c_1,c_1)+(-1)^{n+l_1}K_{n-1}^{(l_1,l_1)}(c_1,-c_1)\}}\Bigg].
\end{aligned}$$

Hence after simplification we find

$$P_n^{\mu,\nu}(x) = P_n(x) + \mu \begin{vmatrix} P_n(x) & K_{n-1}^{(0,l_1)}(x,c_1) + (-1)^{n+l_1} K_{n-1}^{(0,l_1)}(x,-c_1) \\ P_n^{(l_1)}(c_1) & K_{n-1}^{(l_1,l_1)}(c_1,c_1) + (-1)^{n+l_1} K_{n-1}^{(l_1,l_1)}(c_1,-c_1) \end{vmatrix}$$

$$+\nu \begin{vmatrix} P_n(x) & K_{n-1}^{(0,l_2)}(x,c_2) + (-1)^{n+l_2} K_{n-1}^{(0,l_2)}(x,-c_2) \\ P_n^{(l_2)}(c_2) & K_{n-1}^{(l_2,l_2)}(c_2,c_2) + (-1)^{n+l_2} K_{n-1}^{(l_2,l_2)}(c_2,-c_2) \end{vmatrix}$$

$$+\mu\nu \begin{vmatrix} P_n(x) & K_{n-1}^{(0,l_1)}(x,c_1) + (-1)^{n+l_1} K_{n-1}^{(0,l_1)}(x,-c_1) & K_{n-1}^{(0,l_2)}(x,c_2) + (-1)^{n+l_2} K_{n-1}^{(0,l_2)}(x,-c_2) \\ P_n^{(l_1)}(c_1) & K_{n-1}^{(l_1,l_1)}(c_1,c_1) + (-1)^{n+l_1} K_{n-1}^{(l_1,l_1)}(c_1,-c_1) & K_{n-1}^{(l_1,l_2)}(c_1,c_2) + (-1)^{n+l_2} K_{n-1}^{(l_1,l_2)}(c_1,-c_2) \\ P_n^{(l_2)}(c_2) & K_{n-1}^{(l_2,l_1)}(c_2,c_1) + (-1)^{n+l_1} K_{n-1}^{(l_2,l_1)}(c_2,-c_1) & K_{n-1}^{(l_2,l_2)}(c_2,c_2) + (-1)^{n+l_2} K_{n-1}^{(l_2,l_2)}(c_2,-c_2) \end{vmatrix}.$$

We write for $n \in \{0,1,2,\ldots\}$

$$P_n^{\mu,\nu}(x) = P_n(x) + \mu Q_n(x) + \nu R_n(x) + \mu\nu S_n(x) \tag{22}$$

with (4),

$$R_n(x) = \sum_{k=0}^{n} r_{nk} P_k(x), \tag{23}$$

$$S_n(x) = \sum_{k=0}^{n} s_{nk} P_k(x). \tag{24}$$

In this case we have

$$q_{nn} = K_{n-1}^{(l_1,l_1)}(c_1,c_1) + (-1)^{n+l_1} K_{n-1}^{(l_1,l_1)}(c_1,-c_1),$$

$$r_{nn} = K_{n-1}^{(l_2,l_2)}(c_2,c_2) + (-1)^{n+l_2} K_{n-1}^{(l_2,l_2)}(c_2,-c_2),$$

$$s_{nn} = \begin{vmatrix} K_{n-1}^{(l_1,l_1)}(c_1,c_1) + (-1)^{n+l_1} K_{n-1}^{(l_1,l_1)}(c_1,-c_1) & K_{n-1}^{(l_1,l_2)}(c_1,c_2) + (-1)^{n+l_2} K_{n-1}^{(l_1,l_2)}(c_1,-c_2) \\ K_{n-1}^{(l_2,l_1)}(c_2,c_1) + (-1)^{n+l_1} K_{n-1}^{(l_2,l_1)}(c_2,-c_1) & K_{n-1}^{(l_2,l_2)}(c_2,c_2) + (-1)^{n+l_2} K_{n-1}^{(l_2,l_2)}(c_2,-c_2) \end{vmatrix}.$$

Observe that for any choice of c_1 and c_2 it follows that $Q_n(x)=0$ for all $n \in \{0,1,\ldots,l_1+1\}$, $R_n(x)=0$ for all $n \in \{0,1,\ldots,l_2+1\}$ and $S_n(x)=0$ for all $n \in \{0,1,\ldots,l_2+1\}$. Further we have $q_{nn} > 0$ for all $n \in \{l_1+2, l_1+3, l_1+4,\ldots\}$ and $r_{nn} > 0$ for all $n \in \{l_2+2, l_2+3, l_2+4,\ldots\}$. Moreover, by the Cauchy–Schwarz inequality it follows that $s_{nn} > 0$ for all $n \in \{l_2+2, l_2+3, l_2+4,\ldots\}$ unless $l_2 = l_1$ or $l_2 = l_1 + 1$.

If $l_2 = l_1$ (we have assumed $c_1 \neq c_2$ in that case), then $S_{l_2+2}(x) = 0, S_{l_2+3}(x) = 0$ and $s_{nn} > 0$ for all $n \in \{l_2+4, l_2+5, l_2+6,\ldots\}$.

If $l_2 = l_1 + 1$, then $S_{l_2+2}(x) = 0$ and $s_{nn} > 0$ for all $n \in \{l_2+3, l_2+4, l_2+5,\ldots\}$, except in the particular case $c_1 = \sqrt{3}c_2$, since then $S_{l_2+4}(x) = 0$.

3.2. The operators

3.2.1. The general idea

Let $\{P_n(x)\}_{n=0}^{\infty}$ be a system of orthogonal polynomials as introduced in Section 2.1. We will construct linear differential operators, depending linearly on μ and v, hence of the form

$$\boldsymbol{L} + \mu\boldsymbol{A} + v\boldsymbol{B} + \mu v\boldsymbol{C} \tag{25}$$

for which the polynomials $\{P_n^{\mu,v}(x)\}_{n=0}^{\infty}$, found in the preceding section, are eigenfunctions and with sequences of eigenvalues of the form

$$\{\lambda_n + \mu\alpha_n + v\beta_n + \mu v\gamma_n\}_{n=0}^{\infty}. \tag{26}$$

Here $\boldsymbol{A}$ is of the form (8), $\boldsymbol{B}$ and $\boldsymbol{C}$ are of similar form. The general idea of this construction is as follows. By first taking the symmetric (l_1, c_1, μ) perturbation of $\{P_n(x)\}_{n=0}^{\infty}$ and the symmetric (l_2, c_2, v) perturbation of the resulting orthogonal polynomials afterwards we obtain a set V of sequences of eigenvalues of the form (26), where each element of V corresponds to a linear differential operator, depending linearly on v. In the second construction we first take the symmetric (l_2, c_2, v) perturbation of $\{P_n(x)\}_{n=0}^{\infty}$ and the symmetric (l_1, c_1, μ) perturbation of the resulting orthogonal polynomials afterwards. We then obtain a set W of sequences of eigenvalues of the form (26), where each element of W corresponds to a linear differential operator, depending linearly on μ. By [1], Lemma 3.1, it is clear that a sequence of polynomials $\{\pi_n(x)\}_{n=0}^{\infty}$ (with $\deg[\pi_n(x)] = n$) and a sequence of eigenvalues uniquely determine a linear differential operator, mostly of infinite order. Thus to each element of $V \cap W$ corresponds a linear differential operator of the form (25). Since we are only interested in elements of $V \cap W$, we will in the cases that in both constructions certain values of α_n, β_n or γ_n can be chosen arbitrarily, eventually take them the same in both constructions.

Remark 1. *It is sufficient to show that the polynomials $\{P_n^{\mu,v}(x)\}_{n=0}^{\infty}$ are eigenfunctions of linear differential operators of the form* (25) *with eigenvalues of the form* (26) *for all but a countable set of values of $\mu \geqslant 0$, $v \geqslant 0$, since all the expressions we obtain are polynomial identities in μ and v, which in that case will be valid for all values of $\mu \geqslant 0$ and $v \geqslant 0$.*

3.2.2. The first construction

In Section 2.2 we have seen that if

$$P_n^{(l_1)}(c_1) \neq 0 \quad \text{for all } n \in \{l_1, l_1+1, l_1+2, \ldots\} \tag{27}$$

holds, then the polynomials $\{P_n^{\mu}(x)\}_{n=0}^{\infty} = \{P_n(x) + \mu Q_n(x)\}_{n=0}^{\infty}$, orthogonal with respect to

$$\phi_1(p,q) = \langle \sigma, pq \rangle + \mu[p^{(l_1)}(c_1)q^{(l_1)}(c_1) + p^{(l_1)}(-c_1)q^{(l_1)}(-c_1)],$$

are eigenfunctions of operators of the form $\boldsymbol{L} + \mu\boldsymbol{A}$ with eigenvalues $\{\lambda_n + \mu\alpha_n\}_{n=0}^{\infty}$, where $\alpha_0 = 0$, $\{\alpha_n\}_{n=1}^{l_1+1}$ are arbitrary and the numbers $\{\alpha_n\}_{n=l_1+2}^{\infty}$ can be found with (we use the notation (4))

$$\alpha_{l_1+2s} = \alpha_{l_1} + \sum_{j=1}^{s} (\lambda_{l_1+2j} - \lambda_{l_1+2j-2}) q_{l_1+2j\, l_1+2j}, \tag{28}$$

$$\alpha_{l_1+2s+1} = \alpha_{l_1+1} + \sum_{j=1}^{s} (\lambda_{l_1+2j+1} - \lambda_{l_1+2j-1}) q_{l_1+2j+1\, l_1+2j+1}, \tag{29}$$

for $s \in \{1,2,3,\dots\}$. Similarly, if

$$P_n^{(l_2)}(c_2) \neq 0 \quad \text{for all } n \in \{l_2, l_2+1, l_2+2, \dots\} \tag{30}$$

holds, then

$$P_n^{(l_2)}(c_2) + \mu Q_n^{(l_2)}(c_2) \neq 0 \quad \text{for all } n \in \{l_2, l_2+1, l_2+2, \dots\} \tag{31}$$

holds for all but a countable set of values of μ and the polynomials

$$\{P_n^{\mu,\nu}(x)\}_{n=0}^{\infty} = \{(P_n(x) + \mu Q_n(x)) + \nu(R_n(x) + \mu S_n(x))\}_{n=0}^{\infty},$$

orthogonal with respect to (16), are eigenfunctions of operators of the form $\boldsymbol{L} + \mu \boldsymbol{A} + \nu \boldsymbol{B}(\mu)$ with eigenvalues $\{\lambda_n + \mu\alpha_n + \nu\beta_n(\mu)\}_{n=0}^{\infty}$, where $\beta_0(\mu) = 0$, $\{\beta_n(\mu)\}_{n=1}^{l_2+1}$ can be chosen arbitrarily and for $s \in \{1,2,3,\dots\}$

$$\begin{aligned}\beta_{l_2+2s}(\mu) = \beta_{l_2}(\mu) + \sum_{j=1}^{s} (\lambda_{l_2+2j} + \mu\alpha_{l_2+2j} - \lambda_{l_2+2j-2} - \mu\alpha_{l_2+2j-2})\\ \times\{G_{l_2+2j-1}^{(l_2,l_2)}(c_2, c_2; \mu) + G_{l_2+2j-1}^{(l_2,l_2)}(c_2, -c_2; \mu)\},\end{aligned}$$

$$\begin{aligned}\beta_{l_2+2s+1}(\mu) = \beta_{l_2+1}(\mu) + \sum_{j=1}^{s} (\lambda_{l_2+2j+1} + \mu\alpha_{l_2+2j+1} - \lambda_{l_2+2j-1} - \mu\alpha_{l_2+2j-1})\\ \times\{G_{l_2+2j}^{(l_2,l_2)}(c_2, c_2; \mu) - G_{l_2+2j}^{(l_2,l_2)}(c_2, -c_2; \mu)\}.\end{aligned}$$

Here $G_n(x, y; \mu)$ is given by (18). From (19) it easily follows that for $n \in \{1,2,3,\dots\}$

$$(1 + \mu q_{nn})(G_{n-1}^{(l_2,l_2)}(c_2, c_2; \mu) + (-1)^{n+l_2} G_{n-1}^{(l_2,l_2)}(c_2, -c_2; \mu)) = r_{nn} + \mu s_{nn}. \tag{32}$$

We choose $\beta_j(\mu) = \beta_j + \mu\gamma_j$ for $j \in \{1,2,\dots,l_2+1\}$, hence linear in μ, and by using (32), (28) and (29) we obtain for $s \in \{1,2,3,\dots\}$

$$\beta_{l_2+2s}(\mu) = \beta_{l_2} + \mu\gamma_{l_2} + \sum_{j=1}^{s} (\lambda_{l_2+2j} - \lambda_{l_2+2j-2})(r_{l_2+2j\, l_2+2j} + \mu s_{l_2+2j\, l_2+2j}),$$

$$\beta_{l_2+2s+1}(\mu) = \beta_{l_2+1} + \mu\gamma_{l_2+1} + \sum_{j=1}^{s} (\lambda_{l_2+2j+1} - \lambda_{l_2+2j-1})(r_{l_2+2j+1\, l_2+2j+1} + \mu s_{l_2+2j+1\, l_2+2j+1}).$$

It follows that if (27) and (31) hold, then the eigenvalues of the operators $\boldsymbol{L} + \mu\boldsymbol{A} + \nu\boldsymbol{B}(\mu)$, which are linear in ν, can be written in the form (26), where $\alpha_0 = \beta_0 = \gamma_0 = 0$, $\{\alpha_n\}_{n=1}^{l_1+1}$ are arbitrary, the numbers $\{\alpha_n\}_{n=l_1+2}^{\infty}$ can be found with (28) and (29), $\{\beta_n\}_{n=1}^{l_2+1}$ are arbitrary and for $s \in \{1,2,3,\dots\}$

$$\beta_{l_2+2s} = \beta_{l_2} + \sum_{j=1}^{s} (\lambda_{l_2+2j} - \lambda_{l_2+2j-2}) r_{l_2+2j\, l_2+2j}, \tag{33}$$

$$\beta_{l_2+2s+1} = \beta_{l_2+1} + \sum_{j=1}^{s} (\lambda_{l_2+2j+1} - \lambda_{l_2+2j-1}) r_{l_2+2j+1\, l_2+2j+1}, \tag{34}$$

$\{\gamma_n\}_{n=1}^{l_2+1}$ are arbitrary and for $s \in \{1,2,3,\ldots\}$

$$\gamma_{l_2+2s} = \gamma_{l_2} + \sum_{j=1}^{s} (\lambda_{l_2+2j} - \lambda_{l_2+2j-2}) s_{l_2+2j\, l_2+2j}, \tag{35}$$

$$\gamma_{l_2+2s+1} = \gamma_{l_2+1} + \sum_{j=1}^{s} (\lambda_{l_2+2j+1} - \lambda_{l_2+2j-1}) s_{l_2+2j+1\, l_2+2j+1}. \tag{36}$$

3.2.3. The second construction

If (30) holds, then the polynomials $\{P_n^{*v}(x)\}_{n=0}^{\infty} = \{P_n(x) + vR_n(x)\}_{n=0}^{\infty}$, orthogonal with respect to

$$\phi_1^*(p,q) = \langle \sigma, pq \rangle + v[p^{(l_2)}(c_2)q^{(l_2)}(c_2) + p^{(l_2)}(-c_2)q^{(l_2)}(-c_2)],$$

are eigenfunctions of operators of the form $\boldsymbol{L} + v\boldsymbol{B}^*$ with eigenvalues $\{\lambda_n + v\beta_n\}_{n=0}^{\infty}$, where $\beta_0 = 0$, $\{\beta_n\}_{n=1}^{l_2+1}$ are arbitrary and for $\{\beta_n\}_{n=l_2+2}^{\infty}$ we have (33) and (34). Similarly if (27) holds, then

$$P_n^{(l_1)}(c_1) + vR_n^{(l_1)}(c_1) \neq 0 \quad \text{for all } n \in \{l_1, l_1+1, l_1+2, \ldots\} \tag{37}$$

holds for all but a countable set of values of v and the polynomials

$$\{P_n^{\mu,v}(x)\}_{n=0}^{\infty} = \{(P_n(x) + vR_n(x)) + \mu(Q_n(x) + vS_n(x))\}_{n=0}^{\infty},$$

orthogonal with respect to (16), are eigenfunctions of operators of the form $\boldsymbol{L} + v\boldsymbol{B}^* + \mu\boldsymbol{A}^*(v)$ with eigenvalues $\{\lambda_n + v\beta_n + \mu\alpha_n^*(v)\}_{n=0}^{\infty}$, where $\alpha_0^*(v) = 0$, $\{\alpha_n^*(v)\}_{n=1}^{l_1+1}$ can be chosen arbitrarily, and for $t \in \{1,2,3,\ldots\}$

$$\alpha_{l_1+2t}^*(v) = \alpha_{l_1}^*(v) + \sum_{j=1}^{t} (\lambda_{l_1+2j} + v\beta_{l_1+2j} - \lambda_{l_1+2j-2} - v\beta_{l_1+2j-2})$$
$$\times \{G_{l_1+2j-1}^{*(l_1,l_1)}(c_1,c_1;v) + G_{l_1+2j-1}^{*(l_1,l_1)}(c_1,-c_1;v)\},$$

$$\alpha_{l_1+2t+1}^*(v) = \alpha_{l_1+1}^*(v) + \sum_{j=1}^{t} (\lambda_{l_1+2j+1} + v\beta_{l_1+2j+1} - \lambda_{l_1+2j-1} - v\beta_{l_1+2j-1})$$
$$\times \{G_{l_1+2j}^{*(l_1,l_1)}(c_1,c_1;v) - G_{l_1+2j}^{*(l_1,l_1)}(c_1,-c_1;v)\}.$$

Here $G_n^*(x,y;v)$ is given by

$$G_n^*(x,y;v) = \sum_{i=0}^{n} \frac{P_i^{*v}(x)P_i^{*v}(y)}{\phi_1^*(P_i^{*v}, P_i^{*v})}.$$

For $G_n^*(x,y;v)$, analogously to (32), we have for $n \in \{1,2,3,\ldots\}$

$$(1 + vr_{nn})(G_{n-1}^{*(l_1,l_1)}(c_1,c_1;v) + (-1)^{n+l_1} G_{n-1}^{*(l_1,l_1)}(c_1,-c_1;v)) = q_{nn} + vs_{nn}. \tag{38}$$

We choose $\alpha_j^*(v) = \alpha_j + v\gamma_j$ for $j \in \{1,2,\ldots,l_1+1\}$, hence linear in v, and $\{\alpha_j\}_{j=1}^{l_1+1}$, $\{\gamma_j\}_{j=1}^{l_1+1}$ arbitrary. We put $t_0 = \lfloor \frac{l_2-l_1}{2} \rfloor$ and $t_1 = \lfloor \frac{l_2-l_1+1}{2} \rfloor$. Then in the case $l_1 < l_2$ we find for $t \in \{1,2,\ldots,t_1\}$

$$\alpha_{l_1+2t}^*(v) = \alpha_{l_1} + v\gamma_{l_1} + \sum_{j=1}^{t} (\lambda_{l_1+2j} + v\beta_{l_1+2j} - \lambda_{l_1+2j-2} - v\beta_{l_1+2j-2}) q_{l_1+2j\, l_1+2j}$$

and for $t \in \{1,2,\dots,t_0\}$ (if $l_2 > l_1 + 1$)

$$\alpha^*_{l_1+2t+1}(v) = \alpha_{l_1+1} + v\gamma_{l_1+1} + \sum_{j=1}^{t} (\lambda_{l_1+2j+1} + v\beta_{l_1+2j+1} - \lambda_{l_1+2j-1} - v\beta_{l_1+2j-1}) q_{l_1+2j+1\, l_1+2j+1}.$$

This implies that $\alpha^*_j(v) = \alpha_j + v\gamma_j$ for $j \leqslant l_2 + 1$. By (38), (33) and (34) we obtain for $t \in \{t_1 + 1, t_1 + 2, t_1 + 3, \dots\}$

$$\alpha^*_{l_1+2t}(v) = \alpha_{l_1+2t_1} + v\gamma_{l_1+2t_1} + \sum_{j=t_1+1}^{t} (\lambda_{l_1+2j} - \lambda_{l_1+2j-2})(q_{l_1+2j\, l_1+2j} + vs_{l_1+2j\, l_1+2j})$$

and for $t \in \{t_0 + 1, t_0 + 2, t_0 + 3, \dots\}$

$$\alpha^*_{l_1+2t+1}(v) = \alpha_{l_1+2t_0+1} + v\gamma_{l_1+2t_0+1} + \sum_{j=t_0+1}^{t} (\lambda_{l_1+2j+1} - \lambda_{l_1+2j-1})(q_{l_1+2j+1\, l_1+2j+1} + vs_{l_1+2j+1\, l_1+2j+1}).$$

It follows that if (30) and (37) hold, then the eigenvalues of the operators $\boldsymbol{L} + v\boldsymbol{B}^* + \mu\boldsymbol{A}^*(v)$, which are linear in μ, can be written in the form (26), where $\alpha_0 = \beta_0 = \gamma_0 = 0$, $\{\alpha_n\}_{n=1}^{l_1+1}$ are arbitrary, $\{\alpha_n\}_{n=l_1+2}^{\infty}$ can be found with (28) and (29), $\{\beta_n\}_{n=1}^{l_2+1}$ are arbitrary and $\{\beta_n\}_{n=l_2+2}^{\infty}$ can be found with (33) and (34), $\{\gamma_n\}_{n=1}^{l_1+1}$ are arbitrary, for $t \in \{1,2,\dots,t_1\}$ (if $l_2 > l_1$)

$$\gamma_{l_1+2t} = \gamma_{l_1} + \sum_{j=1}^{t} (\beta_{l_1+2j} - \beta_{l_1+2j-2}) q_{l_1+2j\, l_1+2j}, \tag{39}$$

and for $t \in \{1,2,\dots,t_0\}$ (if $l_2 > l_1 + 1$)

$$\gamma_{l_1+2t+1} = \gamma_{l_1+1} + \sum_{j=1}^{t} (\beta_{l_1+2j+1} - \beta_{l_1+2j-1}) q_{l_1+2j+1\, l_1+2j+1}, \tag{40}$$

for $t \in \{t_1 + 1, t_1 + 2, t_1 + 3, \dots\}$

$$\gamma_{l_1+2t} = \gamma_{l_1+2t_1} + \sum_{j=t_1+1}^{t} (\lambda_{l_1+2j} - \lambda_{l_1+2j-2}) s_{l_1+2j\, l_1+2j} \tag{41}$$

and for $t \in \{t_0 + 1, t_0 + 2, t_0 + 3, \dots\}$

$$\gamma_{l_1+2t+1} = \gamma_{l_1+2t_0+1} + \sum_{j=t_0+1}^{t} (\lambda_{l_1+2j+1} - \lambda_{l_1+2j-1}) s_{l_1+2j+1\, l_1+2j+1}. \tag{42}$$

3.2.4. Conclusion

Let the conditions (27) and (30) hold. We see that if the arbitrary values $\{\alpha_n\}_{n=1}^{l_1+1}$ and $\{\beta_n\}_{n=1}^{l_2+1}$ are chosen the same in both constructions, then the values of $\{\alpha_n\}_{n=l_1+2}^{\infty}$ and $\{\beta_n\}_{n=l_2+2}^{\infty}$ become the same in both constructions. In the first construction the values $\{\gamma_n\}_{n=1}^{l_2+1}$ are arbitrary, in the second $\{\gamma_n\}_{n=1}^{l_1+1}$, whereas if $l_1 < l_2$, then the values $\{\gamma_n\}_{n=l_1+2}^{l_2+1}$ are given by (39) and (40). It is clear that the second construction imposes the strongest conditions. In order to see what happens with the higher values of γ_n we consider two cases.

Case 1: *$l_2 - l_1$ is even.* In this case $l_2 = l_1 + 2t_0 = l_1 + 2t_1$ and we see that relation (35) coincides with relation (41) with $t = s + t_1$ and also that relation (36) coincides with relation (42) with $t = s + t_0$.

Case 2: $l_2 - l_1$ *is odd.* In this case $l_2 = l_1 + 2t_0 + 1$ and $l_2 + 1 = l_1 + 2t_1$ and we see that relation (35) coincides with relation (42) with $t = s + t_0$ and also that relation (36) coincides with relation (41) with $t = s + t_1$.

We may conclude that if the eigenvalues are taken as in the second construction, then the corresponding differential operators will depend linearly on μ and ν, hence they will be of the form (25).

4. Two special perturbations

4.1. Representation of the polynomials

The polynomials $\{P_n^{\mu,\nu}(x)\}_{n=0}^{\infty}$, which are orthogonal with respect to

$$\psi_2(p,q) = \langle \sigma, pq \rangle + \mu p^{(l_1)}(0)q^{(l_1)}(0) + \nu p^{(l_2)}(0)q^{(l_2)}(0), \tag{43}$$

where $p, q \in \mathfrak{P}$, $\mu > 0$, $\nu > 0$, $l_1, l_2 \in \{0, 1, 2, \ldots\}$, can be found as a special case of those derived in [3]. Without loss of the generality we may assume

$$l_1 < l_2.$$

We obtain

$$P_n^{\mu,\nu}(x) = P_n(x) + \mu \begin{vmatrix} P_n(x) & K_{n-1}^{(0,l_1)}(x,0) \\ P_n^{(l_1)}(0) & K_{n-1}^{(l_1,l_1)}(0,0) \end{vmatrix} + \nu \begin{vmatrix} P_n(x) & K_{n-1}^{(0,l_2)}(x,0) \\ P_n^{(l_2)}(0) & K_{n-1}^{(l_2,l_2)}(0,0) \end{vmatrix}$$
$$+\mu\nu \begin{vmatrix} P_n(x) & K_{n-1}^{(0,l_1)}(x,0) & K_{n-1}^{(0,l_2)}(x,0) \\ P_n^{(l_1)}(0) & K_{n-1}^{(l_1,l_1)}(0,0) & K_{n-1}^{(l_1,l_2)}(0,0) \\ P_n^{(l_2)}(0) & K_{n-1}^{(l_2,l_1)}(0,0) & K_{n-1}^{(l_2,l_2)}(0,0) \end{vmatrix}.$$

Again for $n \in \{0, 1, 2, \ldots\}$ we use the notation (22) with (4), (23) and (24). Here

$$q_{nn} = K_{n-1}^{(l_1,l_1)}(0,0),$$

$$r_{nn} = K_{n-1}^{(l_2,l_2)}(0,0),$$

$$s_{nn} = \begin{vmatrix} K_{n-1}^{(l_1,l_1)}(0,0) & K_{n-1}^{(l_1,l_2)}(0,0) \\ K_{n-1}^{(l_2,l_1)}(0,0) & K_{n-1}^{(l_2,l_2)}(0,0) \end{vmatrix}.$$

Note that $Q_n(x) = 0$ for all $n \in \{0, 1, \ldots, l_1\}$, $R_n(x) = 0$ for all $n \in \{0, 1, \ldots, l_2\}$ and $S_n(x) = 0$ for all $n \in \{0, 1, \ldots, l_2\}$. We have $q_{nn} > 0$ for all $n \in \{l_1 + 1, l_1 + 2, l_1 + 3, \ldots\}$ and $r_{nn} > 0$ for all $n \in \{l_2 + 1, l_2 + 2, l_2 + 3, \ldots\}$. Moreover, by the Cauchy-Schwarz inequality, it follows that $s_{nn} > 0$ for all $n \in \{l_2 + 1, l_2 + 2, l_2 + 3, \ldots\}$.

Further it is important to note that in the case that $l_2 - l_1$ is odd we have $K_{n-1}^{(l_1,l_2)}(0,0) = 0$, which implies

$$s_{nn} = q_{nn}r_{nn} \quad \text{for all } n \in \{0, 1, 2, \ldots\}. \tag{44}$$

4.2. The operators

Let $\{P_n(x)\}_{n=0}^{\infty}$ be a system of orthogonal polynomials as introduced in Section 2.1. We will construct linear differential operators of the form (25), for which the polynomials $\{P_n^{\mu,\nu}(x)\}_{n=0}^{\infty}$, found in the preceding section, are eigenfunctions. The general idea of this construction is the same as in the preceding case. In the following we will put

$$s_0 = \left\lfloor \frac{l_2 - l_1}{2} \right\rfloor.$$

4.2.1. The first construction

As mentioned in Section 2.3, if

$$P_n^{(l_1)}(0) \neq 0 \quad \text{for all } n \geqslant l_1 \text{ with } n - l_1 \text{ even} \tag{45}$$

holds, then the polynomials $\{P_n^{\mu}(x)\}_{n=0}^{\infty} = \{P_n(x) + \mu Q_n(x)\}_{n=0}^{\infty}$, orthogonal with respect to

$$\phi_2(p,q) = \langle \sigma, pq \rangle + \mu p^{(l_1)}(0) q^{(l_1)}(0),$$

are eigenfunctions of operators of the form $\boldsymbol{L} + \mu \boldsymbol{A}$ with eigenvalues $\{\lambda_n + \mu\alpha_n\}_{n=0}^{\infty}$, where $\alpha_0 = 0$, $\{\alpha_n\}_{n=1}^{l_1}$ (if $l_1 > 0$) and $\{\alpha_{l_1+2s-1}\}_{s=1}^{\infty}$ are arbitrary and $\{\alpha_{l_1+2s}\}_{s=1}^{\infty}$ are given by

$$\alpha_{l_1+2s} = \alpha_{l_1} + \sum_{j=1}^{s} (\lambda_{l_1+2j} - \lambda_{l_1+2j-2}) q_{l_1+2j\, l_1+2j}. \tag{46}$$

Similarly, if

$$P_n^{(l_2)}(0) \neq 0 \quad \text{for all } n \geqslant l_2 \text{ with } n - l_2 \text{ even,} \tag{47}$$

holds, then

$$P_n^{(l_2)}(0) + \mu Q_n^{(l_2)}(0) \neq 0 \quad \text{for all } n \geqslant l_2 \text{ with } n - l_2 \text{ even} \tag{48}$$

holds for all but a countable set of values of μ and the polynomials

$$\{P_n^{\mu,\nu}(x)\}_{n=0}^{\infty} = \{(P_n(x) + \mu Q_n(x)) + \nu(R_n(x) + \mu S_n(x))\}_{n=0}^{\infty},$$

orthogonal with respect to (43), are eigenfunctions of operators of the form $\boldsymbol{L} + \mu\boldsymbol{A} + \nu\boldsymbol{B}(\mu)$ with eigenvalues $\{\lambda_n + \mu\alpha_n + \nu\beta_n(\mu)\}_{n=0}^{\infty}$, where $\beta_0(\mu) = 0$, $\{\beta_n(\mu)\}_{n=1}^{l_2}$ and $\{\beta_{l_2+2s-1}(\mu)\}_{s=1}^{\infty}$ can be chosen arbitrarily, and for $s \in \{1,2,3,\dots\}$

$$\beta_{l_2+2s}(\mu) = \beta_{l_2}(\mu) + \sum_{j=1}^{s} (\lambda_{l_2+2j} + \mu\alpha_{l_2+2j} - \lambda_{l_2+2j-2} - \mu\alpha_{l_2+2j-2}) G_{l_2+2j-1}^{(l_2,l_2)}(0,0;\mu). \tag{49}$$

Here $G_n(x,y;\mu)$ is given by

$$G_n(x,y;\mu) = \sum_{i=0}^{n} \frac{P_i^{\mu}(x) P_i^{\mu}(y)}{\phi_2(P_i^{\mu}, P_i^{\mu})}.$$

By [8] Proposition 3.2 we have for $n \in \{1,2,3,\dots\}$

$$(1 + \mu q_{nn}) G_{n-1}^{(l_2,l_2)}(0,0;\mu) = r_{nn} + \mu s_{nn}. \tag{50}$$

We choose $\beta_j(\mu)=\beta_j+\mu\gamma_j$ for $j\in\{1,2,\ldots,l_2\}$ and $\beta_{l_2+2s-1}(\mu)=\beta_{l_2+2s-1}+\mu\gamma_{l_2+2s-1}$ ($s\in\{1,2,3,\ldots\}$), hence linear in μ. Now let (45) and (48) hold. Then we consider two different cases:

Case 1: l_2-l_1 is even. By (50) and (46) we find for $s\in\{1,2,3,\ldots\}$

$$\beta_{l_2+2s}(\mu)=\beta_{l_2}+\mu\gamma_{l_2}+\sum_{j=1}^{s}(\lambda_{l_2+2j}-\lambda_{l_2+2j-2})(r_{l_2+2j\,l_2+2j}+\mu s_{l_2+2j\,l_2+2j}).$$

It follows that the eigenvalues of the operators $\boldsymbol{L}+\mu\boldsymbol{A}+\nu\boldsymbol{B}(\mu)$, which are linear in ν, can be written in the form (26), where $\alpha_0=\beta_0=\gamma_0=0$, $\{\alpha_n\}_{n=1}^{l_1}$ (if $l_1>0$) and $\{\alpha_{l_1+2s-1}\}_{s=1}^{\infty}$ are arbitrary, $\{\alpha_{l_1+2s}\}_{s=1}^{\infty}$ are given by (46), $\{\beta_n\}_{n=1}^{l_2}$ and $\{\beta_{l_2+2s-1}\}_{s=1}^{\infty}$ are arbitrary, $\{\beta_{l_2+2s}\}_{s=1}^{\infty}$ are given by

$$\beta_{l_2+2s}=\beta_{l_2}+\sum_{j=1}^{s}(\lambda_{l_2+2j}-\lambda_{l_2+2j-2})r_{l_2+2j\,l_2+2j}, \tag{51}$$

$\{\gamma_n\}_{n=1}^{l_2}$ and $\{\gamma_{l_2+2s-1}\}_{s=1}^{\infty}$ are arbitrary, and $\{\gamma_{l_2+2s}\}_{s=1}^{\infty}$ are given by

$$\gamma_{l_2+2s}=\gamma_{l_2}+\sum_{j=1}^{s}(\lambda_{l_2+2j}-\lambda_{l_2+2j-2})s_{l_2+2j\,l_2+2j}. \tag{52}$$

Case 2: l_2-l_1 is odd. By (50) and (44) we find for $s\in\{1,2,3,\ldots\}$

$$\beta_{l_2+2s}(\mu)=\beta_{l_2}+\mu\gamma_{l_2}+\sum_{j=1}^{s}(\lambda_{l_2+2j}+\mu\alpha_{l_2+2j}-\lambda_{l_2+2j-2}-\mu\alpha_{l_2+2j-2})r_{l_2+2j\,l_2+2j}.$$

The values $\alpha_{l_2+2j}=\alpha_{l_1+2s_0+2j+1}$ for $j\in\{0,1,2,\ldots\}$ are arbitrary and we see that for any choice this will result in eigenvalues which are linear in μ. It follows that the eigenvalues of the operators $\boldsymbol{L}+\mu\boldsymbol{A}+\nu\boldsymbol{B}(\mu)$, which are linear in ν, can be written in the form (26), where $\alpha_0=\beta_0=\gamma_0=0$, $\{\alpha_n\}_{n=1}^{l_1}$ (if $l_1>0$) and $\{\alpha_{l_1+2s-1}\}_{s=1}^{\infty}$ are arbitrary, $\{\alpha_{l_1+2s}\}_{s=1}^{\infty}$ are given by (46), $\{\beta_n\}_{n=1}^{l_2}$ and $\{\beta_{l_2+2s-1}\}_{s=1}^{\infty}$ are arbitrary, $\{\beta_{l_2+2s}\}_{s=1}^{\infty}$ are given by (51), $\{\gamma_n\}_{n=1}^{l_2}$ and $\{\gamma_{l_2+2s-1}\}_{s=1}^{\infty}$ are arbitrary and $\{\gamma_{l_2+2s}\}_{s=1}^{\infty}$ are given by

$$\gamma_{l_2+2s}=\gamma_{l_2}+\sum_{j=1}^{s}(\alpha_{l_2+2j}-\alpha_{l_2+2j-2})r_{l_2+2j\,l_2+2j}. \tag{53}$$

4.2.2. The second construction

If (47) holds, then the polynomials $\{P_n^{*\nu}(x)\}_{n=0}^{\infty}=\{P_n(x)+\nu R_n(x)\}_{n=0}^{\infty}$, orthogonal with respect to

$$\phi_2^*(p,q)=\langle\sigma,pq\rangle+\nu p^{(l_2)}(0)q^{(l_2)}(0),$$

are eigenfunctions of operators of the form $\boldsymbol{L}+\nu\boldsymbol{B}^*$ with eigenvalues $\{\lambda_n+\nu\beta_n\}_{n=0}^{\infty}$, where $\beta_0=0$, $\{\beta_n\}_{n=1}^{l_2}$ and $\{\beta_{l_2+2s-1}\}_{s=1}^{\infty}$ are arbitrary and $\{\beta_{l_2+2s}\}_{s=1}^{\infty}$ are given by (51). Similarly, if (45) holds, then

$$P_n^{(l_1)}(0)+\nu R_n^{(l_1)}(0)\neq 0\quad\text{for all } n\geqslant l_1 \text{ with } n-l_1 \text{ even} \tag{54}$$

holds for all but a countable set of values of ν and the polynomials

$$\{P_n^{\mu,\nu}(x)\}_{n=0}^{\infty}=\{(P_n(x)+\nu R_n(x))+\mu(Q_n(x)+\nu S_n(x))\}_{n=0}^{\infty},$$

orthogonal with respect to (43), are eigenfunctions of operators of the form $\boldsymbol{L}+v\boldsymbol{B}^*+\mu\boldsymbol{A}^*(v)$ with eigenvalues $\{\lambda_n+v\beta_n+\mu\alpha_n^*(v)\}_{n=0}^{\infty}$, where $\alpha_0^*(v)=0$, $\{\alpha_n^*(v)\}_{n=1}^{l_1}$ (if $l_1>0$) and $\{\alpha_{l_1+2s-1}^*(v)\}_{s=1}^{\infty}$ can be chosen arbitrarily and $\{\alpha_{l_1+2s}^*(v)\}_{s=1}^{\infty}$ are given by

$$\alpha_{l_1+2s}^*(v)=\alpha_{l_1}^*(v)+\sum_{j=1}^{s}(\lambda_{l_1+2j}+v\beta_{l_1+2j}-\lambda_{l_1+2j-2}-v\beta_{l_1+2j-2})G_{l_1+2j-1}^{*(l_1,l_1)}(0,0;v). \tag{55}$$

Here

$$G_n^*(x,y;v)=\sum_{i=0}^{n}\frac{P_i^{*v}(x)P_i^{*v}(y)}{\phi_2^*(P_i^{*v},P_i^{*v})}.$$

For $G_n^*(x,y;v)$ analogously to (50) we have for $n\in\{1,2,3,\ldots\}$

$$(1+vr_{nn})G_{n-1}^{*(l_1,l_1)}(0,0;v)=q_{nn}+vs_{nn}. \tag{56}$$

We choose $\alpha_j^*(v)=\alpha_j+v\gamma_j$ for $j\in\{1,2,\ldots,l_1\}$ (if $l_1>0$) and $\alpha_{l_1+2s-1}^*(v)=\alpha_{l_1+2s-1}+v\gamma_{l_1+2s-1}$ for $s\in\{1,2,3,\ldots\}$, hence linear in v, and $\{\alpha_j\}_{j=1}^{l_1}$, $\{\gamma_j\}_{j=1}^{l_1}$, $\{\alpha_{l_1+2s-1}\}_{s=1}^{\infty}$, $\{\gamma_{l_1+2s-1}\}_{s=1}^{\infty}$ are arbitrary. Now let (47) and (54) hold. We consider two different cases:

Case 1: *l_2-l_1 is even.* Then we find for $s\in\{1,2,\ldots,s_0\}$

$$\alpha_{l_1+2s}^*(v)=\alpha_{l_1}+v\gamma_{l_1}+\sum_{j=1}^{s}(\lambda_{l_1+2j}+v\beta_{l_1+2j}-\lambda_{l_1+2j-2}-v\beta_{l_1+2j-2})q_{l_1+2j\ \ l_1+2j} \tag{57}$$

and for $s\in\{1,2,3,\ldots\}$ (since $l_2=l_1+2s_0$)

$$\alpha_{l_2+2s}^*(v)=\alpha_{l_1}+\sum_{j=1}^{s+s_0}(\lambda_{l_1+2j}-\lambda_{l_1+2j-2})q_{l_1+2j\ \ l_1+2j}+v\left(\gamma_{l_2}+\sum_{j=1}^{s}(\lambda_{l_2+2j}-\lambda_{l_2+2j-2})s_{l_2+2j\ \ l_2+2j}\right).$$

It follows that the eigenvalues of the operators $\boldsymbol{L}+v\boldsymbol{B}^*+\mu\boldsymbol{A}^*(v)$, which are linear in μ, can be written in the form (26), where $\alpha_0=\beta_0=\gamma_0=0, \{\alpha_n\}_{n=1}^{l_1}$ (if $l_1>0$) and $\{\alpha_{l_1+2s-1}\}_{s=1}^{\infty}$ are arbitrary, $\{\alpha_{l_1+2s}\}_{s=1}^{\infty}$ are given by (46), $\{\beta_n\}_{n=1}^{l_2}$ and $\{\beta_{l_2+2s-1}\}_{s=1}^{\infty}$ are arbitrary, $\{\beta_{l_2+2s}\}_{s=1}^{\infty}$ are given by (51), $\{\gamma_n\}_{n=1}^{l_1}$ (if $l_1>0$) and $\{\gamma_{l_1+2s-1}\}_{s=1}^{\infty}$ are arbitrary, $\{\gamma_{l_1+2s}\}_{s=1}^{s_0}$ are given by

$$\gamma_{l_1+2s}=\gamma_{l_1}+\sum_{j=1}^{s}(\beta_{l_1+2j}-\beta_{l_1+2j-2})q_{l_1+2j\ \ l_1+2j} \tag{58}$$

and $\{\gamma_{l_2+2s}\}_{s=1}^{\infty}$ are given by (52).

Case 2: *l_2-l_1 is odd.* We find, using (55) and the fact that in this case $G_{n-1}^{*(l_1,l_1)}(0,0;v)=q_{nn}$ for all $n\in\{1,2,3,\ldots\}$, that $\{\alpha_{l_1+2s}\}_{s=1}^{\infty}$ are given by (46) and that $\{\gamma_{l_1+2s}\}_{s=1}^{\infty}$ are given by (58). It follows that the eigenvalues of the operators $\boldsymbol{L}+v\boldsymbol{B}^*+\mu\boldsymbol{A}^*(v)$, which are linear in μ, can be written in the form (26), where $\alpha_0=\beta_0=\gamma_0=0, \{\alpha_n\}_{n=1}^{l_1}$ (if $l_1>0$) and $\{\alpha_{l_1+2s-1}\}_{s=1}^{\infty}$ are arbitrary, $\{\alpha_{l_1+2s}\}_{s=1}^{\infty}$ are given by (46), $\{\beta_n\}_{n=1}^{l_2}$ and $\{\beta_{l_2+2s-1}\}_{s=1}^{\infty}$ are arbitrary and $\{\beta_{l_2+2s}\}_{s=1}^{\infty}$ are given by (51), $\{\gamma_n\}_{n=1}^{l_1}$ (if $l_1>0$) and $\{\gamma_{l_1+2s-1}\}_{s=1}^{\infty}$ are arbitrary, $\{\gamma_{l_1+2s}\}_{s=1}^{\infty}$ are given by (58).

4.2.3. Conclusion

Let the conditions (45), (47) hold. We see that if the arbitrary numbers $\{\alpha_n\}_{n=1}^{l_1}$ (if $l_1>0$), $\{\alpha_{l_1+2s-1}\}_{s=1}^{\infty}$, $\{\beta_n\}_{n=1}^{l_2}$ and $\{\beta_{l_2+2s-1}\}_{s=1}^{\infty}$ are chosen the same in both constructions, then in both constructions: $\{\alpha_{l_1+2s}\}_{s=1}^{\infty}$ are given by (46) and $\{\beta_{l_2+2s}\}_{s=1}^{\infty}$ by (51).

Case 1: *l_2-l_1 is even.* In the first construction the values $\{\gamma_n\}_{n=1}^{l_2}$ and $\{\gamma_{l_2+2s-1}\}_{s=1}^{\infty}$ are arbitrary, in the second the values $\{\gamma_n\}_{n=1}^{l_1}$ (if $l_1>0$) and $\{\gamma_{l_1+2s-1}\}_{s=1}^{\infty}$ are arbitrary and the numbers $\{\gamma_{l_1+2s}\}_{s=1}^{s_0}$ are given by (58). Further in both constructions: $\{\gamma_{l_2+2s}\}_{s=1}^{\infty}$ are given by (52). It is clear that the second construction imposes the strongest conditions.

We may conclude that if the eigenvalues $\{\gamma_n\}_{n=1}^{\infty}$ are taken as in the second construction, then the corresponding differential operators will depend linearly on μ and v, hence they will be of the form (25).

Case 2: *l_2-l_1 is odd.* In the first construction the values $\{\gamma_n\}_{n=1}^{l_2}$ and $\{\gamma_{l_2+2s-1}\}_{s=1}^{\infty}$ are arbitrary and $\{\gamma_{l_2+2s}\}_{s=1}^{\infty}$ are given by (53). In the second construction the values $\{\gamma_n\}_{n=1}^{l_1}$ (if $l_1>0$) and $\{\gamma_{l_1+2s+1}\}_{s=0}^{\infty}$ are arbitrary and $\{\gamma_{l_1+2s}\}_{s=1}^{\infty}$ are given by (58).

We may conclude that if the eigenvalues $\{\gamma_n\}_{n=1}^{\infty}$ are taken as $\{\gamma_n\}_{n=1}^{l_1}$ (if $l_1>0$) and $\{\gamma_{l_1+2s+1}\}_{s=0}^{s_0}$ arbitrary, $\{\gamma_{l_2+2s}\}_{s=1}^{\infty}$ by (53) and $\{\gamma_{l_1+2s}\}_{s=1}^{\infty}$ by (58), then the corresponding differential operators will depend linearly on μ and v, hence they will be of the form (25).

5. A symmetric and a special perturbation

5.1. Representation of the polynomials

In this section we consider the polynomials $\{P_n^{\mu,v}(x)\}_{n=0}^{\infty}$, orthogonal with respect to the symmetric bilinear form (of Sobolev type) defined by

$$\psi_3(p,q)=\langle\sigma,pq\rangle+\mu[p^{(l_1)}(c)q^{(l_1)}(c)+p^{(l_1)}(-c)q^{(l_1)}(-c)]+vp^{(l_2)}(0)q^{(l_2)}(0), \tag{59}$$

where $p,q\in\mathfrak{P}$, $\mu>0, v>0, l_1,l_2\in\{0,1,2,\ldots\}, c>0$.

We construct the polynomials orthogonal with respect to ψ_3 by first taking the symmetric (l_1,c,μ) perturbation of $\{P_n(x)\}_{n=0}^{\infty}$, thus obtaining the polynomials $\{P_n^{\mu}(x)\}_{n=0}^{\infty}$, orthogonal with respect to

$$\phi_3(p,q)=\langle\sigma,pq\rangle+\mu[p^{(l_1)}(c)q^{(l_1)}(c)+p^{(l_1)}(-c)q^{(l_1)}(-c)].$$

They can be written as (see Section 2.2)

$$P_n^{\mu}(x)=P_n(x)+\mu\begin{vmatrix} P_n(x) & K_{n-1}^{(0,l_1)}(x,c)+(-1)^{n+l_1}K_{n-1}^{(0,l_1)}(x,-c)\\ P_n^{(l_1)}(c) & K_{n-1}^{(l_1,l_1)}(c,c)+(-1)^{n+l_1}K_{n-1}^{(l_1,l_1)}(c,-c)\end{vmatrix}. \tag{60}$$

Afterwards the special (l_2,v) perturbation of $\{P_n^{\mu}(x)\}_{n=0}^{\infty}$ is taken, leading to the polynomials $\{P_n^{\mu,v}(x)\}_{n=0}^{\infty}$, for which (see Section 2.3)

$$P_n^{\mu,v}(x)=(1+vG_{n-1}^{(l_2,l_2)}(0,0;\mu))P_n^{\mu}(x)-vP_n^{\mu(l_2)}(0)G_{n-1}^{(0,l_2)}(x,0;\mu).$$

Here $G_n(x,y;\mu)$ is given by

$$G_n(x,y;\mu)=\sum_{i=0}^{n}\frac{P_i^{\mu}(x)P_i^{\mu}(y)}{\phi_3(P_i^{\mu},P_i^{\mu})}. \tag{61}$$

1. If $n-l_2$ is odd, then $K_{n-1}^{(l_2,l_1)}(0,c)+(-1)^{n+l_1}K_{n-1}^{(l_2,l_1)}(0,-c)=0$ and $P_n^{(l_2)}(0)=0$, which implies that we can put

$$P_n^{\mu,v}(x)=CP_n^{\mu}(x)$$

for any normalization constant C. We choose

$$C = 1 + \nu K_{n-1}^{(l_2,l_2)}(0,0).$$

2. If $n - l_2$ is even, then we proceed as follows. We differentiate formula (19) l_2 times with respect to y and put $c_1 = c$, thus obtaining

$$\begin{aligned}
&G_n^{(0,l_2)}(x,y;\mu) + (-1)^{m+l_2} G_n^{(0,l_2)}(x,-y;\mu)\\
&\quad = K_n^{(0,l_2)}(x,y) + (-1)^{m+l_2} K_n^{(0,l_2)}(x,-y)\\
&\qquad - \mu \frac{\{K_n^{(0,l_1)}(x,c) + (-1)^{m+l_1} K_n^{(0,l_1)}(x,-c)\}\{K_n^{(l_1,l_2)}(c,y) + (-1)^{m+l_2} K_n^{(l_1,l_2)}(c,-y)\}}{1 + \mu\{K_n^{(l_1,l_1)}(c,c) + (-1)^{m+l_1} K_n^{(l_1,l_1)}(c,-c)\}}.
\end{aligned}$$

Further we put $m = l_2, y = 0$ and divide by 2, finding

$$G_n^{(0,l_2)}(x,0;\mu) = K_n^{(0,l_2)}(x,0) - \mu \frac{\{K_n^{(0,l_1)}(x,c) + (-1)^{l_1+l_2} K_n^{(0,l_1)}(x,-c)\} K_n^{(l_1,l_2)}(c,0)}{1 + \mu\{K_n^{(l_1,l_1)}(c,c) + (-1)^{l_1+l_2} K_n^{(l_1,l_1)}(c,-c)\}}. \tag{62}$$

Hence

$$\begin{aligned}
P_n^{\mu,\nu}(x) = &\left[1 + \nu K_{n-1}^{(l_2,l_2)}(0,0) - \mu\nu \frac{\{K_{n-1}^{(l_2,l_1)}(0,c) + (-1)^{l_1+l_2} K_{n-1}^{(l_2,l_1)}(0,-c)\} K_{n-1}^{(l_1,l_2)}(c,0)}{1 + \mu\{K_{n-1}^{(l_1,l_1)}(c,c) + (-1)^{l_1+l_2} K_{n-1}^{(l_1,l_1)}(c,-c)\}}\right]\\
&\times[(1 + \mu\{K_{n-1}^{(l_1,l_1)}(c,c) + (-1)^{n+l_1} K_{n-1}^{(l_1,l_1)}(c,-c)\})P_n(x)\\
&-\mu P_n^{(l_1)}(c)\{K_{n-1}^{(0,l_1)}(x,c) + (-1)^{n+l_1} K_{n-1}^{(0,l_1)}(x,-c)\}]\\
&-\nu[(1 + \mu\{K_{n-1}^{(l_1,l_1)}(c,c) + (-1)^{n+l_1} K_{n-1}^{(l_1,l_1)}(c,-c)\})P_n^{(l_2)}(0)\\
&-\mu P_n^{(l_1)}(c)\{K_{n-1}^{(l_2,l_1)}(0,c) + (-1)^{n+l_1} K_{n-1}^{(l_2,l_1)}(0,-c)\}]\\
&\times \left[K_{n-1}^{(0,l_2)}(x,0) - \mu \frac{\{K_{n-1}^{(0,l_1)}(x,c) + (-1)^{l_1+l_2} K_{n-1}^{(0,l_1)}(x,-c)\} K_{n-1}^{(l_1,l_2)}(c,0)}{1 + \mu\{K_{n-1}^{(l_1,l_1)}(c,c) + (-1)^{l_1+l_2} K_{n-1}^{(l_1,l_1)}(c,-c)\}}\right].
\end{aligned}$$

In the case we consider $(-1)^{l_1+l_2} = (-1)^{l_1+n}$, which results in

$$\begin{aligned}
P_n^{\mu,\nu}(x) = P_n(x) &+ \mu \begin{vmatrix} P_n(x) & K_{n-1}^{(0,l_1)}(x,c) + (-1)^{n+l_1} K_{n-1}^{(0,l_1)}(x,-c) \\ P_n^{(l_1)}(c) & K_{n-1}^{(l_1,l_1)}(c,c) + (-1)^{n+l_1} K_{n-1}^{(l_1,l_1)}(c,-c) \end{vmatrix} + \nu \begin{vmatrix} P_n(x) & K_{n-1}^{(0,l_2)}(x,0) \\ P_n^{(l_2)}(0) & K_{n-1}^{(l_2,l_2)}(0,0) \end{vmatrix}\\
&+ \mu\nu \begin{vmatrix} P_n(x) & K_{n-1}^{(0,l_1)}(x,c) + (-1)^{n+l_1} K_{n-1}^{(0,l_1)}(x,-c) & K_{n-1}^{(0,l_2)}(x,0) \\ P_n^{(l_1)}(c) & K_{n-1}^{(l_1,l_1)}(c,c) + (-1)^{n+l_1} K_{n-1}^{(l_1,l_1)}(c,-c) & K_{n-1}^{(l_1,l_2)}(c,0) \\ P_n^{(l_2)}(0) & K_{n-1}^{(l_2,l_1)}(0,c) + (-1)^{n+l_1} K_{n-1}^{(l_2,l_1)}(0,-c) & K_{n-1}^{(l_2,l_2)}(0,0) \end{vmatrix}.
\end{aligned} \tag{63}$$

By our choice of the normalization constant C in the case that $n-l_2$ is odd, formula (63) is valid for all $n\in\{0,1,2,\ldots\}$ and we use the notation (22) with (4), (23) and (24). Here

$$q_{nn}=K_{n-1}^{(l_1,l_1)}(c,c)+(-1)^{n+l_1}K_{n-1}^{(l_1,l_1)}(c,-c),$$

$$r_{nn}=K_{n-1}^{(l_2,l_2)}(0,0),$$

$$s_{nn}=\begin{vmatrix} K_{n-1}^{(l_1,l_1)}(c,c)+(-1)^{n+l_1}K_{n-1}^{(l_1,l_1)}(c,-c) & K_{n-1}^{(l_1,l_2)}(c,0)\\ K_{n-1}^{(l_2,l_1)}(0,c)+(-1)^{n+l_1}K_{n-1}^{(l_2,l_1)}(0,-c) & K_{n-1}^{(l_2,l_2)}(0,0)\end{vmatrix}.$$

We have

$$s_{nn}=q_{nn}r_{nn} \quad \text{for all } n \text{ with } n-l_2 \text{ is odd.} \tag{64}$$

Note that $Q_n(x)=0$ for all $n\in\{0,1,\ldots,l_1+1\}$, $R_n(x)=0$ for all $n\in\{0,1,\ldots,l_2\}$ and $S_n(x)=0$ for all $n\in\{0,1,\ldots,\max\{l_1+1,l_2\}\}$. We have $q_{nn}>0$ for all $n\in\{l_1+2,l_1+3,l_1+4,\ldots\}$ and $r_{nn}>0$ for all $n\in\{l_2+1,l_2+2,l_2+3,\ldots\}$. Moreover, by the Cauchy–Schwarz inequality and (64) it follows that $s_{nn}>0$ for all $n\in\{\max\{l_1+1,l_2\}+1,\max\{l_1+1,l_2\}+2,\ldots\}$ unless $l_2=l_1$ or $l_2=l_1+1$. If $l_2=l_1$, then $S_{l_2+2}(x)=0$ and $s_{nn}>0$ for all $n\in\{l_2+3,l_2+4,l_2+5,\ldots\}$. If $l_2=l_1+1$, then $S_{l_2+2}(x)=0$ and $s_{nn}>0$ for all $n\in\{l_2+1,l_2+3,l_2+4,l_2+5,\ldots\}$.

5.2. The operators

Let $\{P_n(x)\}_{n=0}^{\infty}$ be a system of orthogonal polynomials as introduced in Section 2.1. We will construct linear differential operators of the form (25), for which the polynomials $\{P_n^{\mu,\nu}(x)\}_{n=0}^{\infty}$, found in the preceding section, are eigenfunctions. The general idea of this construction is the same as in the preceding cases. We put

$$s_1=\left\lfloor\frac{|l_2-l_1|+1}{2}\right\rfloor.$$

5.2.1. The first construction

In Section 2.2 it was shown that if

$$P_n^{(l_1)}(c)\neq 0 \quad \text{for all } n\in\{l_1,l_1+1,l_1+2,\ldots\} \tag{65}$$

holds, then the polynomials $\{P_n^{\mu}(x)\}_{n=0}^{\infty}=\{P_n(x)+\mu Q_n(x)\}_{n=0}^{\infty}$, orthogonal with respect to

$$\phi_3(p,q)=\langle\sigma,pq\rangle+\mu[p^{(l_1)}(c)q^{(l_1)}(c)+p^{(l_1)}(-c)q^{(l_1)}(-c)],$$

are eigenfunctions of operators of the form $\boldsymbol{L}+\mu\boldsymbol{A}$ with eigenvalues $\{\lambda_n+\mu\alpha_n\}_{n=0}^{\infty}$, where $\alpha_0=0,\{\alpha_n\}_{n=1}^{l_1+1}$ are arbitrary and the numbers $\{\alpha_n\}_{n=l_1+2}^{\infty}$ can be found with

$$\alpha_{l_1+2s}=\alpha_{l_1}+\sum_{j=1}^{s}(\lambda_{l_1+2j}-\lambda_{l_1+2j-2})q_{l_1+2j\ \ l_1+2j}, \tag{66}$$

$$\alpha_{l_1+2s+1}=\alpha_{l_1+1}+\sum_{j=1}^{s}(\lambda_{l_1+2j+1}-\lambda_{l_1+2j-1})q_{l_1+2j+1\ \ l_1+2j+1}, \tag{67}$$

for $s \in \{1,2,3,\dots\}$. By Section 2.3, if

$$P_n^{(l_2)}(0) \neq 0 \quad \text{for all } n \geqslant l_2 \text{ with } n-l_2 \text{ even} \tag{68}$$

holds, then

$$P_n^{(l_2)}(0) + \mu Q_n^{(l_2)}(0) \neq 0 \quad \text{for all } n \geqslant l_2 \text{ with } n-l_2 \text{ even} \tag{69}$$

holds for all but a countable set of values of μ and the polynomials

$$\{P_n^{\mu,\nu}(x)\}_{n=0}^{\infty} = \{(P_n(x)+\mu Q_n(x)) + \nu(R_n(x)+\mu S_n(x))\}_{n=0}^{\infty},$$

orthogonal with respect to (59), are eigenfunctions of operators of the form $\boldsymbol{L}+\mu\boldsymbol{A}+\nu\boldsymbol{B}(\mu$ with eigenvalues $\{\lambda_n+\mu\alpha_n+\nu\beta_n(\mu)\}_{n=0}^{\infty}$, where $\beta_0(\mu)=0, \{\beta_n(\mu)\}_{n=1}^{l_2}$ (if $l_2>0$) and $\{\beta_{l_2+2s-1}(\mu)\}_{s=1}^{\infty}$ can be chosen arbitrarily, and for $s \in \{1,2,3,\dots\}$

$$\beta_{l_2+2s}(\mu) = \beta_{l_2}(\mu) + \sum_{j=1}^{s} (\lambda_{l_2+2j}+\mu\alpha_{l_2+2j}-\lambda_{l_2+2j-2}-\mu\alpha_{l_2+2j-2}) G_{l_2+2j-1}^{(l_2,l_2)}(0,0;\mu). \tag{70}$$

Here $G_n(x,y;\mu)$ is given by (61). From (62) it easily follows that for $n-l_2$ even

$$(1+\mu q_{nn}) G_{n-1}^{(l_2,l_2)}(0,0;\mu) = r_{nn} + \mu s_{nn}. \tag{71}$$

We choose $\beta_j(\mu)=\beta_j+\mu\gamma_j$ for $j \in \{1,2,\dots,l_2\}$ (if $l_2>0$) and $\beta_{l_2+2s-1}(\mu)=\beta_{l_2+2s-1}+\mu\gamma_{l_2+2s-1}$ ($s \in \{1,2,3,\dots\}$), hence linear in μ. Now let (65) and (69) hold. Then we consider two different cases:

Case 1: $l_2 \geqslant l_1$. By (70), (66), (67) and (71) we find for $s \in \{1,2,3,\dots\}$

$$\beta_{l_2+2s}(\mu) = \beta_{l_2} + \mu\gamma_{l_2} + \sum_{j=1}^{s} (\lambda_{l_2+2j}-\lambda_{l_2+2j-2})(r_{l_2+2j\ l_2+2j} + \mu s_{l_2+2j\ l_2+2j}).$$

It follows that the eigenvalues of the operators $\boldsymbol{L}+\mu\boldsymbol{A}+\nu\boldsymbol{B}(\mu)$, which are linear in ν, can be written in the form (26), where $\alpha_0=\beta_0=\gamma_0=0, \{\alpha_n\}_{n=1}^{l_1+1}$ are arbitrary and the numbers $\{\alpha_n\}_{n=l_1+2}^{\infty}$ are given by (66) and (67), $\{\beta_n\}_{n=1}^{l_2}$ (if $l_2>0$) and $\{\beta_{l_2+2s-1}\}_{s=1}^{\infty}$ are arbitrary, $\{\beta_{l_2+2s}\}_{s=1}^{\infty}$ are given by

$$\beta_{l_2+2s} = \beta_{l_2} + \sum_{j=1}^{s} (\lambda_{l_2+2j}-\lambda_{l_2+2j-2}) r_{l_2+2j\ l_2+2j}, \tag{72}$$

$\{\gamma_n\}_{n=1}^{l_2}$ (if $l_2>0$) and $\{\gamma_{l_2+2s-1}\}_{s=1}^{\infty}$ are arbitrary, and $\{\gamma_{l_2+2s}\}_{s=1}^{\infty}$ are given by

$$\gamma_{l_2+2s} = \gamma_{l_2} + \sum_{j=1}^{s} (\lambda_{l_2+2j}-\lambda_{l_2+2j-2}) s_{l_2+2j\ l_2+2j}. \tag{73}$$

Case 2: $l_1 > l_2$. Since for $1 \leqslant j \leqslant s_1$ we have $q_{l_2+2j\ l_2+2j} = s_{l_2+2j\ l_2+2j} = 0$, by (70) and (71) it follows that for $s \in \{1,2,\dots,s_1\}$

$$\beta_{l_2+2s}(\mu) = \beta_{l_2} + \mu\gamma_{l_2} + \sum_{j=1}^{s} (\lambda_{l_2+2j}+\mu\alpha_{l_2+2j}-\lambda_{l_2+2j-2}-\mu\alpha_{l_2+2j-2}) r_{l_2+2j\ l_2+2j}.$$

By (70), (66), (67) and (71) we find for $s \in \{s_1+1, s_1+2, s_1+3,\dots\}$

$$\beta_{l_2+2s}(\mu) = \beta_{l_2+2s_1} + \mu\gamma_{l_2+2s_1} + \sum_{j=s_1+1}^{s} (\lambda_{l_2+2j}-\lambda_{l_2+2j-2})(r_{l_2+2j\ l_2+2j} + \mu s_{l_2+2j\ l_2+2j}).$$

It follows that the eigenvalues of the operators $\boldsymbol{L}+\mu\boldsymbol{A}+v\boldsymbol{B}(\mu)$, which are linear in v, can be written in the form (26), where $\alpha_0=\beta_0=\gamma_0=0, \{\alpha_n\}_{n=1}^{l_1+1}$ are arbitrary, $\{\alpha_n\}_{n=l_1+2}^{\infty}$ can be found by (66) and (67), $\{\beta_n\}_{n=1}^{l_2}$ (if $l_2>0$) and $\{\beta_{l_2+2s-1}\}_{s=1}^{\infty}$ are arbitrary, $\{\beta_{l_2+2s}\}_{s=1}^{\infty}$ are given by (72), $\{\gamma_n\}_{n=1}^{l_2}$ (if $l_2>0$) and $\{\gamma_{l_2+2s-1}\}_{s=1}^{\infty}$ are arbitrary. For $s\in\{1,2,\ldots,s_1\}$

$$\gamma_{l_2+2s}=\gamma_{l_2}+\sum_{j=1}^{s}(\alpha_{l_2+2j}-\alpha_{l_2+2j-2})r_{l_2+2j\ l_2+2j} \tag{74}$$

and for $s\in\{s_1+1,s_1+2,s_1+3,\ldots\}$

$$\gamma_{l_2+2s}=\gamma_{l_2+2s_1}+\sum_{j=s_1+1}^{s}(\lambda_{l_2+2j}-\lambda_{l_2+2j-2})s_{l_2+2j\ l_2+2j}. \tag{75}$$

5.2.2. The second construction

By Section 2.3, if (68) holds, then the polynomials $\{P_n^{*v}(x)\}_{n=0}^{\infty}=\{P_n(x)+vR_n(x)\}_{n=0}^{\infty}$, orthogonal with respect to

$$\phi_3^*(p,q)=\langle\sigma,pq\rangle+vp^{(l_2)}(0)q^{(l_2)}(0),$$

are eigenfunctions of operators of the form $\boldsymbol{L}+v\boldsymbol{B}^*$ with eigenvalues $\{\lambda_n+v\beta_n\}_{n=0}^{\infty}$, where $\beta_0=0,\{\beta_n\}_{n=1}^{l_2}$ (if $l_2>0$) and $\{\beta_{l_2+2s-1}\}_{s=1}^{\infty}$ are arbitrary and $\{\beta_{l_2+2s}\}_{s=1}^{\infty}$ are given by (72). By Section 2.2, if (65) holds, then

$$P_n^{(l_1)}(c)+vR_n^{(l_1)}(c)\neq 0 \quad \text{for all } n\in\{l_1,l_1+1,l_1+2,\ldots\} \tag{76}$$

holds for all but a countable set of values of v and the polynomials

$$\{P_n^{\mu,v}(x)\}_{n=0}^{\infty}=\{(P_n(x)+vR_n(x))+\mu(Q_n(x)+vS_n(x))\}_{n=0}^{\infty},$$

orthogonal with respect to (59), are eigenfunctions of operators of the form $\boldsymbol{L}+v\boldsymbol{B}^*+\mu\boldsymbol{A}^*(v)$ with eigenvalues $\{\lambda_n+v\beta_n+\mu\alpha_n^*(v)\}_{n=0}^{\infty}$, where $\alpha_0^*(v)=0,\{\alpha_n^*(v)\}_{n=1}^{l_1+1}$ can be chosen arbitrarily and

$$\alpha_{l_1+2s}^*(v)=\alpha_{l_1}^*(v)+\sum_{j=1}^{s}(\lambda_{l_1+2j}+v\beta_{l_1+2j}-\lambda_{l_1+2j-2}-v\beta_{l_1+2j-2})\times\{G_{l_1+2j-1}^{*(l_1,l_1)}(c,c;v)+G_{l_1+2j-1}^{*(l_1,l_1)}(c,-c;v)\}, \tag{77}$$

$$\alpha_{l_1+2s+1}^*(v)=\alpha_{l_1+1}^*(v)+\sum_{j=1}^{s}(\lambda_{l_1+2j+1}+v\beta_{l_1+2j+1}-\lambda_{l_1+2j-1}-v\beta_{l_1+2j-1})\times\{G_{l_1+2j}^{*(l_1,l_1)}(c,c;v)-G_{l_1+2j}^{*(l_1,l_1)}(c,-c;v)\} \tag{78}$$

for $s\in\{1,2,3,\ldots\}$. Here for $G_n^*(x,y;v)$ we have

$$G_n^*(x,y;v)=\sum_{i=0}^{n}\frac{P_i^{*v}(x)P_i^{*v}(y)}{\phi_3^*(P_i^{*v},P_i^{*v})}.$$

For $G_n^*(x,y;v)$, by [8], Proposition 3.2, we have for $n\in\{1,2,3,\ldots\}$

$$(1+vr_{nn})\{G_{n-1}^{*(l_1,l_1)}(c,c;v)+(-1)^{n+l_1}G_{n-1}^{*(l_1,l_1)}(c,-c;v)\}=q_{nn}+vs_{nn}. \tag{79}$$

We choose $\alpha_j^*(v)=\alpha_j+v\gamma_j$ for $j\in\{1,2,\ldots,l_1+1\}$, hence linear in v, and $\{\alpha_j\}_{j=1}^{l_1+1},\{\gamma_j\}_{j=1}^{l_1+1}$ arbitrary. Now let (68) and (76) hold. We consider four different cases.

Case 1: $l_2 > l_1 + 1, l_2 - l_1$ *even.* Since for $1 \leqslant j \leqslant s_1$ we have $r_{l_1+2j\ l_1+2j} = s_{l_1+2j\ l_1+2j} = 0$, by (77) and (79) it follows for $s \in \{1, 2, \ldots, s_1\}$ that

$$\alpha^*_{l_1+2s}(v) = \alpha_{l_1} + v\gamma_{l_1} + \sum_{j=1}^{s} (\lambda_{l_1+2j} + v\beta_{l_1+2j} - \lambda_{l_1+2j-2} - v\beta_{l_1+2j-2}) q_{l_1+2j\ l_1+2j}.$$

By (72), (77) and (79) for $s \in \{s_1 + 1, s_1 + 2, s_1 + 3, \ldots\}$

$$\alpha^*_{l_1+2s}(v) = \alpha_{l_1+2s_1} + v\gamma_{l_1+2s_1} + \sum_{j=s_1+1}^{s} (\lambda_{l_1+2j} - \lambda_{l_1+2j-2})(q_{l_1+2j\ l_1+2j} + vs_{l_1+2j\ l_1+2j}).$$

Further by (64), (78) and (79) we have for $s \in \{1, 2, 3, \ldots\}$

$$\alpha^*_{l_1+2s+1}(v) = \alpha_{l_1+1} + v\gamma_{l_1+1} + \sum_{j=1}^{s} (\lambda_{l_1+2j+1} + v\beta_{l_1+2j+1} - \lambda_{l_1+2j-1} - v\beta_{l_1+2j-1}) q_{l_1+2j+1\ l_1+2j+1}.$$

It follows that the eigenvalues of the operators $\boldsymbol{L} + v\boldsymbol{B}^* + \mu\boldsymbol{A}^*(v)$, which are linear in μ, can be written in the form (26), where $\alpha_0 = \beta_0 = \gamma_0 = 0, \{\alpha_n\}_{n=1}^{l_1+1}$ are arbitrary, $\{\alpha_n\}_{n=l_1+2}^{\infty}$ can be found by (66) and (67), $\{\beta_n\}_{n=1}^{l_2}$ and $\{\beta_{l_2+2s-1}\}_{s=1}^{\infty}$ are arbitrary, $\{\beta_{l_2+2s}\}_{s=1}^{\infty}$ are given by (72), $\{\gamma_n\}_{n=1}^{l_1+1}$ are arbitrary, $\{\gamma_{l_1+2s}\}_{s=1}^{s_1}$ are given by

$$\gamma_{l_1+2s} = \gamma_{l_1} + \sum_{j=1}^{s} (\beta_{l_1+2j} - \beta_{l_1+2j-2}) q_{l_1+2j\ l_1+2j}, \tag{80}$$

$\{\gamma_{l_2+2s}\}_{s=1}^{\infty}$ are given by (73) and $\{\gamma_{l_1+2s+1}\}_{s=1}^{\infty}$ are given by

$$\gamma_{l_1+2s+1} = \gamma_{l_1+1} + \sum_{j=1}^{s} (\beta_{l_1+2j+1} - \beta_{l_1+2j-1}) q_{l_1+2j+1\ l_1+2j+1}. \tag{81}$$

Case 2: $l_2 > l_1 + 1, l_2 - l_1$ *odd.* By (64), (77) and (79) we have for $s \in \{1, 2, 3, \ldots\}$

$$\alpha^*_{l_1+2s}(v) = \alpha_{l_1} + v\gamma_{l_1} + \sum_{j=1}^{s} (\lambda_{l_1+2j} + v\beta_{l_1+2j} - \lambda_{l_1+2j-2} - v\beta_{l_1+2j-2}) q_{l_1+2j\ l_1+2j}.$$

Since for $1 \leqslant j \leqslant s_1 - 1$ we have $r_{l_1+2j+1\ l_1+2j+1} = s_{l_1+2j+1\ l_1+2j+1} = 0$, by (78) and (79) it follows for $s \in \{1, 2, \ldots, s_1 - 1\}$ that

$$\alpha^*_{l_1+2s+1}(v) = \alpha_{l_1+1} + v\gamma_{l_1+1} + \sum_{j=1}^{s} (\lambda_{l_1+2j+1} + v\beta_{l_1+2j+1} - \lambda_{l_1+2j-1} - v\beta_{l_1+2j-1}) q_{l_1+2j+1\ l_1+2j+1}.$$

Further by (72), (78) and (79) for $s \in \{s_1, s_1 + 1, s_1 + 2, \ldots\}$

$$\alpha^*_{l_1+2s+1}(v) = \alpha_{l_1+2s_1-1} + v\gamma_{l_1+2s_1-1} + \sum_{j=s_1}^{s} (\lambda_{l_1+2j+1} - \lambda_{l_1+2j-1})(q_{l_1+2j+1\ l_1+2j+1} + vs_{l_1+2j+1\ l_1+2j+1}).$$

It follows that the eigenvalues of the operators $\boldsymbol{L} + v\boldsymbol{B}^* + \mu\boldsymbol{A}^*(v)$, which are linear in μ, can be written in the form (26), where $\alpha_0 = \beta_0 = \gamma_0 = 0, \{\alpha_n\}_{n=1}^{l_1+1}$ are arbitrary, $\{\alpha_n\}_{n=l_1+2}^{\infty}$ can be found by (66) and (67), $\{\beta_n\}_{n=1}^{l_2}$ and $\{\beta_{l_2+2s-1}\}_{s=1}^{\infty}$ are arbitrary, $\{\beta_{l_2+2s}\}_{s=1}^{\infty}$ are given by (72), $\{\gamma_n\}_{n=1}^{l_1+1}$ are arbitrary, $\{\gamma_{l_1+2s}\}_{s=1}^{\infty}$ are given by (80), $\{\gamma_{l_1+2s+1}\}_{s=1}^{s_1-1}$ are given by (81) and $\{\gamma_{l_1+2s_1+2s-1}\}_{s=1}^{\infty} = \{\gamma_{l_2+2s}\}_{s=1}^{\infty}$ are given by (73).

Case 3: $l_1+1\geqslant l_2, l_1-l_2$ *even.* By (72), (77) and (79) for $s\in\{1,2,3,\ldots\}$

$$\alpha^*_{l_1+2s}(v)=\alpha_{l_1}+v\gamma_{l_1}+\sum_{j=1}^{s}(\lambda_{l_1+2j}-\lambda_{l_1+2j-2})(q_{l_1+2j\ l_1+2j}+vs_{l_1+2j\ l_1+2j}).$$

Further by (64), (78) and (79) have for $s\in\{1,2,3,\ldots\}$

$$\alpha^*_{l_1+2s+1}(v)=\alpha_{l_1+1}+v\gamma_{l_1+1}+\sum_{j=1}^{s}(\lambda_{l_1+2j+1}+v\beta_{l_1+2j+1}-\lambda_{l_1+2j-1}-v\beta_{l_1+2j-1})q_{l_1+2j+1\ l_1+2j+1}.$$

It follows that the eigenvalues of the operators $\boldsymbol{L}+v\boldsymbol{B}^*+\mu\boldsymbol{A}^*(v)$, which are linear in μ, can be written in the form (26), where $\alpha_0=\beta_0=\gamma_0=0, \{\alpha_n\}_{n=1}^{l_1+1}$ are arbitrary, $\{\alpha_n\}_{n=l_1+2}^{\infty}$ can be found by (66) and (67), $\{\beta_n\}_{n=1}^{l_2}$ (if $l_2>0$) and $\{\beta_{l_2+2s-1}\}_{s=1}^{\infty}$ are arbitrary, $\{\beta_{l_2+2s}\}_{s=1}^{\infty}$ are given by (72), $\{\gamma_n\}_{n=1}^{l_1+1}$ are arbitrary, $\{\gamma_{l_1+2s}\}_{s=1}^{\infty}$ are given by

$$\gamma_{l_1+2s}=\gamma_{l_1}+\sum_{j=1}^{s}(\lambda_{l_1+2j}-\lambda_{l_1+2j-2})s_{l_1+2j\ l_1+2j} \tag{82}$$

and $\{\gamma_{l_1+2s+1}\}_{s=1}^{\infty}$ by (81).

Case 4: $l_1+1\geqslant l_2$, l_1-l_2 *odd.* By (64), (77) and (79) we have for $s\in\{1,2,3,\ldots\}$

$$\alpha^*_{l_1+2s}(v)=\alpha_{l_1}+v\gamma_{l_1}+\sum_{j=1}^{s}(\lambda_{l_1+2j}+v\beta_{l_1+2j}-\lambda_{l_1+2j-2}-v\beta_{l_1+2j-2})q_{l_1+2j\ l_1+2j}.$$

Further by (72), (78) and (79) for $s\in\{1,2,3,\ldots\}$

$$\alpha^*_{l_1+2s+1}(v)=\alpha_{l_1+1}+v\gamma_{l_1+1}+\sum_{j=1}^{s}(\lambda_{l_1+2j+1}-\lambda_{l_1+2j-1})(q_{l_1+2j+1\ l_1+2j+1}+vs_{l_1+2j+1\ l_1+2j+1}).$$

It follows that the eigenvalues of the operators $\boldsymbol{L}+v\boldsymbol{B}^*+\mu\boldsymbol{A}^*(v)$, which are linear in μ, can be written in the form (26), where $\alpha_0=\beta_0=\gamma_0=0, \{\alpha_n\}_{n=1}^{l_1+1}$ are arbitrary, $\{\alpha_n\}_{n=l_1+2}^{\infty}$ can be found by (66) and (67), $\{\beta_n\}_{n=1}^{l_2}$ (if $l_2>0$) and $\{\beta_{l_2+2s-1}\}_{s=1}^{\infty}$ are arbitrary, $\{\beta_{l_2+2s}\}_{s=1}^{\infty}$ are given by (72), $\{\gamma_n\}_{n=1}^{l_1+1}$ are arbitrary, $\{\gamma_{l_1+2s}\}_{s=1}^{\infty}$ are given by (80) and $\{\gamma_{l_1+2s+1}\}_{s=1}^{\infty}$ by

$$\gamma_{l_1+2s+1}=\gamma_{l_1+1}+\sum_{j=1}^{s}(\lambda_{l_1+2j+1}-\lambda_{l_1+2j-1})s_{l_1+2j+1\ l_1+2j+1}. \tag{83}$$

5.2.3. Conclusion

Let the conditions (65), (68) hold. Note that $s_1=\lfloor\frac{|l_2-l_1|+1}{2}\rfloor$. We see that in both constructions and in all cases $\{\alpha_n\}_{n=1}^{l_1+1}$ are arbitrary, $\{\alpha_{l_1+2s}\}_{s=1}^{\infty}$ are given by (66), $\{\alpha_{l_1+2s+1}\}_{s=1}^{\infty}$ by (67), $\{\beta_n\}_{n=1}^{l_2}$ (if $l_2>0$) and $\{\beta_{l_2+2s-1}\}_{s=1}^{\infty}$ are arbitrary and $\{\beta_{l_2+2s}\}_{s=1}^{\infty}$ are given by (72). For the conditions on $\{\gamma_n\}_{n=1}^{\infty}$ we have to consider different cases.

Case 1: $l_2>l_1+1$, l_2-l_1 *even.* In this case the second construction imposes the strongest conditions: $\{\gamma_n\}_{n=1}^{l_1+1}$ are arbitrary, $\{\gamma_{l_1+2s}\}_{s=1}^{s_1}$ are given by (80), $\{\gamma_{l_1+2s+1}\}_{s=1}^{\infty}$ by (81) and $\{\gamma_{l_2+2s}\}_{s=1}^{\infty}$ by (73).

Case 2: $l_2 > l_1 + 1$, $l_2 - l_1$ *odd*. In this case the second construction imposes the strongest conditions: $\{\gamma_n\}_{n=1}^{l_1+1}$ are arbitrary, $\{\gamma_{l_1+2s}\}_{s=1}^{\infty}$ are given by (80), $\{\gamma_{l_1+2s+1}\}_{s=1}^{s_1-1}$ by (81) and $\{\gamma_{l_2+2s}\}_{s=1}^{\infty}$ by (73).

Case 3: $l_1 + 1 \geqslant l_2$, $l_1 - l_2$ *even*. In order to obtain eigenvalues which satisfy the conditions of both constructions $\{\gamma_n\}_{n=1}^{l_2}$ (if $l_2 > 0$) and $\{\gamma_{l_2+2s+1}\}_{s=0}^{s_1}$ are arbitrary, $\{\gamma_{l_2+2s}\}_{s=1}^{s_1}$ (if $s_1 > 0$) are given by (74), $\{\gamma_{l_1+2s}\}_{s=1}^{\infty}$ are given by (82) and $\{\gamma_{l_1+2s+1}\}_{s=1}^{\infty}$ are given by (81).

Case 4: $l_1 + 1 > l_2$, $l_1 - l_2$ *odd*. In order to obtain eigenvalues which satisfy the conditions of both constructions $\{\gamma_n\}_{n=1}^{l_2}$ (if $l_2 > 0$) and $\{\gamma_{l_2+2s-1}\}_{s=1}^{s_1}$ are arbitrary, $\{\gamma_{l_2+2s}\}_{s=1}^{s_1}$ are given by (74), $\{\gamma_{l_1+2s}\}_{s=1}^{\infty}$ by (80) and $\{\gamma_{l_1+2s+1}\}_{s=1}^{\infty}$ by (83).

Case 5: $l_1+1=l_2$. In order to obtain eigenvalues which satisfy the conditions of both constructions $\{\gamma_n\}_{n=1}^{l_1+1}$ are arbitrary, $\{\gamma_{l_1+2s}\}_{s=1}^{\infty}$ are given by (80) and $\{\gamma_{l_1+2s+1}\}_{s=1}^{\infty}$ by (83).

We may conclude that if the eigenvalues are taken as indicated, then the corresponding differential operators will depend linearly on μ and ν, hence they will be of the form (25).

6. Conclusion

In the Sections 3–5 three different combinations of two linear perturbations to the orthogonal polynomials with respect to a symmetric positive-definite moment functional are considered, all three resulting in orthogonal polynomials with respect to a new symmetric positive-definite functional (of Sobolev type). In all the cases these orthogonal polynomials turn out to be eigenfunctions of a class of linear differential operators of the form (25) with sequences of eigenvalues of the form (26). This was shown by considering two constructions, one resulting in a set of sequences of eigenvalues V and one resulting in a set of sequences of eigenvalues W. Each element of $V \cap W$ leads to a unique linear differential operator of the form (25).

We found that in all the three cases the values of $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ become the same in both constructions (they actually are the same as in the case of one perturbation of that particular kind), if in both constructions the same choices are made for those values which are arbitrary. The requirements for the values of $\{\gamma_n\}_{n=1}^{\infty}$, however, usually are different for both constructions resulting in specific conditions for these in order to obtain sequences of eigenvalues belonging to $V \cap W$.

7. Examples

7.1. Two symmetric perturbations

Consider the generalization of the Gegenbauer polynomials, polynomials orthogonal with respect to the inner product

$$(p,q) = \frac{\Gamma(2\alpha+2)}{2^{2\alpha+1}\Gamma(\alpha+1)^2}\int_{-1}^{1} p(x)q(x)(1-x^2)^{\alpha}\,dx \\ +M[p^{(l_1)}(-1)q^{(l_1)}(-1)+p^{(l_1)}(1)q^{(l_1)}(1)]+N[p^{(l_2)}(-1)q^{(l_2)}(-1)+p^{(l_2)}(1)q^{(l_2)}(1)], \tag{84}$$

$p,q \in \mathfrak{P}$, $l_2 > l_1 \geqslant 0, M \geqslant 0, N \geqslant 0, \alpha > -1$, which in the case $l_1 = 0, l_2 = 1$ are studied in [5,6]. In the case $l_1 = 0, N = 0$, generalizing earlier results (see [12–16]), Koekoek [11] (see also [9]) showed that the polynomials are eigenfunctions of a class of linear differential operators, of the form $\boldsymbol{L}+M\boldsymbol{A}$ and usually of infinite order, and with eigenvalues of the form $\{\lambda_n + M\alpha_n\}_{n=0}^{\infty}$. Here $\boldsymbol{L}$ denotes the second-order differential operator

$$(x^2-1)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2(\alpha+1)x\frac{\mathrm{d}}{\mathrm{d}x}$$

and $\lambda_n = n(n+2\alpha+1)$ for $n \in \{0,1,2,\ldots\}$. For $M > 0$ this class contains one operator of finite order $2\alpha+4$ if α is a nonnegative integer; for other values of $\alpha > -1$ all operators are of infinite order. By the results of Section 3 we may conclude that the polynomials orthogonal with respect to the inner product (84) are eigenfunctions of a class of linear differential operators of the form $\boldsymbol{L} + M\boldsymbol{A} + N\boldsymbol{B} + MN\boldsymbol{C}$ and for the eigenvalues it is known which can be chosen arbitrarily and how the other eigenvalues can be determined. It is shown in [4] that in the case $l_1 = 0, l_2 = 1$ one of these linear differential operators is of finite order if α is a nonnegative integer and further this operator is given there.

7.2. Two special perturbations

We can also consider another generalization of the Gegenbauer polynomials, polynomials orthogonal with respect to the inner product

$$(p,q) = \frac{\Gamma(2\alpha+2)}{2^{2\alpha+1}\Gamma(\alpha+1)^2}\int_{-1}^{1} p(x)q(x)(1-x^2)^{\alpha}\,\mathrm{d}x + Mp^{(l_1)}(0)q^{(l_1)}(0) + Np^{(l_2)}(0)q^{(l_2)}(0), \quad (85)$$

$p,q \in \mathfrak{P}$, $l_2 > l_1 \geqslant 0, M \geqslant 0, N \geqslant 0, \alpha > -1$. By the results of Section 4 we may conclude that the polynomials orthogonal with respect to the inner product (85) are eigenfunctions of a class of linear differential operators of the form $\boldsymbol{L}+M\boldsymbol{A}+N\boldsymbol{B}+MN\boldsymbol{C}$ and for the eigenvalues it is known which can be chosen arbitrarily and how the other eigenvalues can be determined. In the case $N = 0$, $l_1 = 0$ the operators can be derived without much calculation. In [12] Section 6 the quadratic transformations

$$\frac{P_{2n}^{(\alpha,\alpha)}(x)}{P_{2n}^{(\alpha,\alpha)}(1)} = \frac{P_n^{(\alpha,-\frac{1}{2})}(2x^2-1)}{P_n^{(\alpha,-\frac{1}{2})}(1)}, \qquad \frac{P_{2n+1}^{(\alpha,\alpha)}(x)}{P_{2n+1}^{(\alpha,\alpha)}(1)} = x\frac{P_n^{(\alpha,\frac{1}{2})}(2x^2-1)}{P_n^{(\alpha,\frac{1}{2})}(1)},$$

(see [18], (4.1.5)) are used to find linear differential operators for the Jacobi type polynomials with $\beta = \pm\frac{1}{2}$ and the perturbation only at $x = 1$ from the Gegenbauer polynomials with a symmetric perturbation at $x = \pm 1$ as in the preceding case. Later in [10] the general case for Jacobi type polynomials with perturbations at $x = -1$ and $x = 1$ was studied and from the special case $\beta = \pm\frac{1}{2}$ and the perturbation only at $x = -1$, a case where the differential operator turns out to be of infinite order, we may conclude, using the same quadratic transformations, that in the case $N = 0$, $l_1 = 0$ the polynomials, orthogonal with respect to the inner product (85), are eigenfunctions of a class of linear differential operators which all necessarily are of infinite order, if $M > 0$.

7.3. A symmetric and a special perturbation

We can also consider another generalization of the Gegenbauer polynomials, polynomials orthogonal with respect to the inner product

$$(p,q) = \frac{\Gamma(2\alpha+2)}{2^{2\alpha+1}\Gamma(\alpha+1)^2}\int_{-1}^{1} p(x)q(x)(1-x^2)^\alpha\,\mathrm{d}x$$
$$+M[p^{(l_1)}(-1)q^{(l_1)}(-1) + p^{(l_1)}(1)q^{(l_1)}(1)] + Np^{(l_2)}(0)q^{(l_2)}(0), \tag{86}$$

$p,q \in \mathfrak{P}$, $l_1, l_2 \in \{0,1,2,\ldots\}, M \geqslant 0, N \geqslant 0, \alpha > -1$. By the results of Section 5 we may conclude that the polynomials orthogonal with respect to the inner product (86) are eigenfunctions of a class of linear differential operators of the form $\boldsymbol{L}+M\boldsymbol{A}+N\boldsymbol{B}+MN\boldsymbol{C}$ and for the eigenvalues it is known which can be chosen arbitrarily and how the other eigenvalues can be determined.

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